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# **Beyond the short run: The longer time scale volatility of investment value**

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## **Abstract**

Fund and other investments often exhibit longer run volatility associated with macroeconomic or other dynamics to an extent inconsistent with the efficient market accumulation model. Volatility and performance models or metrics based on one-period returns or simple extensions can fail to pick this up, resulting in suboptimal investment policies, or welfare losses if exit happens to be forced at the wrong time. We show how to use wavelet analysis to resolve problems of detection, attribution and welfare measurement, including assigning volatility metrics and path risk, while dynamic value at risk ideas can be applied to establish clearance points relative to any benchmark comparator path. Generalisations of the spectral utility function can guide investment policy or be used to design optimal portfolios. Band pass portfolios can be designed that smooth investor exposure to long or short run instabilities in investment value.

**Key words:** Band pass portfolios, path risk, portfolio theory, spectral utility functions, long term volatility, wavelets, value at risk.

**JEL classification:** G11, G14-15, G23; E32, E44; C32

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## **I Introduction**

Although many theoretical models of intertemporal portfolio allocation have been devised, the long run is under-represented in practical portfolio theory and its outgrowths such as hedging algorithms. Computation and use of one-period returns remains the predominant raw material for portfolio constructs and construction. It is understandable why this should be the case: one period returns are stationary or nearly so, so they are well adapted for standard statistical measures of reward and variation, which in turn can be easily incorporated into mean-variance portfolios. Extensions such as GARCH or similar volatility modelling can be used to derive hedges that vary with time along with conditional properties of asset returns. But the problem with short run data is that the user can easily become preoccupied with the short term frame of mind. Market or other noise can swamp signals that while weak in the short run are far from weak in the long run. Weak local dependence can be quite consistent with strong global dependence.

When the focus of attention shifts to the longer run, one has to incorporate environmental influences that also act in the longer run: the macroeconomics of business cycles, interest rates or exchange rates, economic policy and structural change. The pricing of a stock, for instance, can be taken as the discounted sum of expected future earnings, and the latter are driven by the business cycle. If the market is fully efficient and far sighted, then even over the longer run the total return index for a stock or market index should follow a canonical model such as an exponential Ito process, in which the drift term incorporates the required rate of return or cost of investor capital, and the volatility term encompasses the revisions in expectations about future earnings and the economy in general. To be sure, macroeconomics can still enter via the cost of capital, reflecting a variable risk premium associated with human capital or other systematic risks. Likewise, the volatility term can incorporate risks associated with the particular stage of the business cycle, or concerns about exchange or interest rate movements. But the model itself remains virtually intact.

In practice, things do not quite work out that way over the longer run, no matter that such models can be adequate approximations over the short run. In the first instance this is an empirical judgement. Many studies on exchange rates have shown that uncovered interest parity does not hold, with or without a risk premium; equivalently, the return to a forward contract is not a martingale difference process.

Likewise some of the longer term patterns of national stock market indices (see below) are hard to reconcile with the above exponential Ito process - it would require a dramatic and unrealistic state-dependent specification of the drift and volatility processes to make them so.

A second difficulty is that the market models, whether theoretical or empirical, must have the flexibility to encompass different information sets. Investors are not all the same: they operate off different models (whether mental or otherwise) of the market and the underlying macroeconomics that drives it. Over time, different investors enter or leave the market, so that the operative information sets themselves follow a dynamics. Attempts have been made in the rational expectations literature to formalise such effects (e.g. Bowden 1988, §5.3). A good way of looking at things is to say that some investors have more detailed knowledge of the market and its dynamics than others, who even in the best (most rational) of worlds, could perceive only the broader brush. A given empirical model should be flexible enough to allow for differences of this kind, or a way of formalising them.

In the longer run therefore, it may be a bad idea to assume automatically that canonical models of value will hold. Empirical work should not assume that they do. In turn, portfolio selection methodology that relies on empirical returns should not assume that they are temporally independent over the longer run. It would make more sense for a passive or long-term 'buy and hold' portfolio to be predicated not so much on one period returns, as on properties of the path as whole. The long-term paths of different assets classes follow quite different dynamic behaviour. It does not automatically follow that the future path will exhibit the same long term volatility pattern, in just the same way that one should not assume in classic mean-variance analysis that future asset return distributions will necessarily mirror the past. Nevertheless there may be a predisposition of some assets to exhibit characteristic long-term volatility patterns, in the same way that business cycles and longer-term exchange rate variation have not yet vanished from world economies. It may be that economic behaviour and economic policy rules have between them a predisposition to cause large-scale volatility, even if precise causal models are hard to evolve. How to measure these patterns is one of the topics to be explored. The way in which they are measured should feed naturally into portfolio selection methodology.

What is needed is a concept of empirical path risk. Path risk is concerned with properties of the path as a whole, including but not limited to terminal values. The

pricing of exotic derivatives is usually path dependent, but in this context, the path risk is commonly thought of as synonymous with model risk and is in any case linked to some presumed data generating model. The task for an empirical path risk concept is to measure the risk of an entire time path without making too many assumptions. In a portfolio context, mean-variance and related problems such as hedging are concerned only with metrics for distributional risk: means, variances and higher order moments relative to some assumed probability distribution. The task for path risk metrics is to assist judgements about how long-term path behaviour of differing asset classes can be measured. Moreover, path risk and measurement should be able to encompass dynamic generalisations of many of the risk metrics in common use: means, variances, extreme values, or value at risk (VaR). Finally, the results should assist the design of portfolios that are superior on the same path behaviour metrics; and once again, should have a correspondence with notions associated with static portfolios such as the mean - variance efficient frontier.

The present paper tackles issues and algorithms associated with path risk. It does so on an empirical level, with the aim of developing practical portfolio solutions when the focus of investment attention is the long run, as with superannuation or long term growth investment funds. The chosen tool is wavelet analysis. This provides an elegant and quite general way of overcoming the restraints of static distributional theory, which are not well adapted to the intertemporal context and require restrictive maintained hypotheses as to the underlying data generation mechanism. One can use the energy decomposition to design portfolios tailored to preferences as between long or short run variation. Band pass portfolios can exclude altogether designated long or short-term value fluctuations.

The scheme of the paper is as follows. Section II reviews wavelet technology for the benefit of readers who may be unfamiliar with it. Illustrative results introduce the application used in the body of the paper. Section III is a closer look at what is required for a practical theory of path risk, including the relationship with value at risk. Section IV contains the bulk of development as it relates to portfolio construction. It is shown how to generalise empirical mean-variance analysis to encompass entire paths rather than just a time series of returns. The performances of the two approaches – wavelet and classic mean variance – are compared. Other points of connection are with spectral utility functions, and to band pass filtering in electronic engineering. There is an application to a small international equities

portfolio with a foreign exchange hedging element. Section V offers some concluding remarks.

## **II Wavelet concepts**

The methodology to be used in the present study draws on wavelet analysis. This is a very flexible way of breaking down a given time series into longer run and shorter run components, with very few maintained assumptions about the process generating the given series. It is in the first instance simply a descriptive tool, but a very powerful one. It overcomes many of the limitations associated with spectral analysis, while preserving the same insights. Readers familiar with spectral analysis will recall that this decomposes a given series into the sum of sinusoids of different frequencies (a process called ‘complex demodulation’). It also attaches amplitudes or power to these sinusoids, so that if one frequency is more powerful than others, much of the variance in the given series can be explained in terms of a well-defined cycle at this frequency. However, the elementary sinusoids themselves do not change over time, either in their frequency or their amplitude. This is one of the limitations of spectral analysis, although from time to time suggestions were made as to how to develop time varying spectra (e.g. Priestley 1965). But even here the change had to be very slow over an extended period of time, or else there was implicit theorising as to the underlying data generation process.

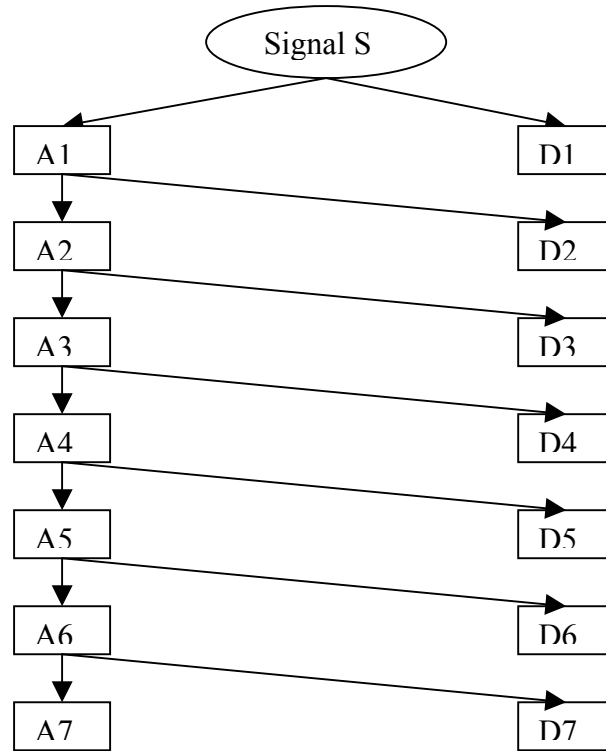
Limitations of such kinds were effectively removed by the development of wavelet theory and practice, notably by later authors such as Mallat (1989), Daubechies (1988, 1990, 1992), Coifman *et al* (1990), Cohen *et al* (1992). For useful reviews of the use of wavelet analysis in economics, see Ramsay (1999), Schleicher (2002), or Crowley (2005).

A wavelet is rather like a sinusoid localised at a particular point in time, so that its power drops off rapidly on either side of that time point. Moving along through time, one fits a succession of such wavelets. Each time point contains contributions from wavelets of the same ‘scale’ (quasi frequency) but centred at neighbouring points. In addition, it will also contain contributions from wavelets of different scales, corresponding to cycles of different frequencies. By a similar mathematical argument to complex demodulation, one can express the series at any point in time as a sum of the wavelets of different scales. The shorter scales represent higher frequency fluctuations, while the large scale wavelets capture the long run movements. A more

detailed account is given in Appendix 1, which also depicts the wavelet family used in the present study, namely the Coif 5 wavelets. Collectively across different scales, the wavelets of either family are flexible enough to allow for asymmetric local cycles of rather arbitrary form, so this is no longer a story requiring regular sinusoidal patterns.

Although all chosen from the same generic family, the wavelets are normalised to refer either to the cycles ('mother wavelets') or long term trend or quasi trends ('father wavelets'). The results of fitting mother (cyclical) wavelets of different scales are called the 'details' ( $D$ ) and they are additive in their effect. Progressive sums, by adding more details, are called the 'approximations' ( $A$ ). Figure 1 is a schematic decomposition. Level 1 is the smallest scale or highest quasi frequency, so  $D_1$  represents the cycle at this highest level of detail. The given series is then split into  $D_1 + A_1$ , where  $A_1$  is the series once the very shortest fluctuations have been removed. Levels 2,3.... contain successively less small-scale complexity. Extracting these leads to broader time frame approximations designed to reveal longer run cycles and ultimately the trend. An 'average period' construct for a given level of detail  $D$  can be derived by finding the sinusoid whose period most closely matches that of the wavelet fitted at any point in time, suitably adjusted for its scale. Then one simply averages out these local equivalent periods over time. This enables us to think of the successive details as corresponding to progressively longer cycles, just as in spectral analysis.

The overall effect is rather like adjusting more and more exactly the focus of a microscope. One of the more celebrated images in wavelet exposition is that of Mme Daubechies' eye as viewed from successively closer up. At long range one sees only the general features, with the rest being blurred. These are like  $D_6$  or  $D_7$  plus the  $A_7$ , although in this case two dimensional. Moving closer, one see higher level details of the iris, ultimately up to  $D_1$ .



**Figure 1: Decomposition into successive details and approximations**

The above analogy is useful, for it gives us a way of thinking about the market efficiency problem. Suppose expectations were unbiased, but with respect to differing information filtrations. Those with poorer information would see only the lower level detail. To such investors the path history would appear as something like the path A6 or A7, surrounded by an band of vagueness representing all the higher level detail, but appearing as a blur, as in the eye analogy. Investors with better information would see greater detail and hence potentially be able to make money at the expense of the average trader. The overall analogy is not exact, because information unfolds in real time whereas the wavelet record is a historical decomposition over a given time interval. Nevertheless, it remains a useful one, for it illustrates that path risk is not absolute: it depends upon who knows what and will not look the same to all investors, even before taking into account their differing personal circumstances or appetites for risk.

As with spectral analysis, one can measure the amplitudes of each detail in the form of the variance of the wavelet detail. Unlike spectral analysis this is a local concept, differing over time. However, one can use compute the average variance over the given time horizon and present the results in the form of a table of average

wavelet energies (AWE) at the different levels of detail. Covariance concepts also exist, more or less corresponding to those of classical statistics.

### *Wavelet decompositions*

Figure 2 illustrates wavelet decompositions for some of the asset classes used in the present study. The object variable (or dependent variable) in each case is the log of the total return index. Further discussion relating to the choice of object variable, including rates of return, appears in section III below.

Table 1 is an AWE decomposition for the asset classes used in the present study using monthly data from Jan 1988 to Dec 2005 (see the Data Appendix for definitions and sources). The maximum detail available for such a data run is level 7. What is left over is taken to be the trend, though it may remain more complex than the standard log linear trend. Indeed this is one of the strengths of wavelet analysis that it makes no prejudgements about the form of any underlying trend or of any underlying stochastic process. The vantage point is that of an international equity portfolio for a New Zealand investor wishing to invest in the US and other major stock markets. The total return stock indexes as given refer to the own currency return, whereas in practice this would be a compound of the own currency return and the relevant exchange rate against the NZ dollar. Such aspects are considered in more detail in section IV, which deals with portfolio formation.

The remaining asset appearing in table 1 is the total return index on a one-month forward contract on the US dollar against the NZ dollar. This corresponds to a portfolio long in zero coupon NZ bonds or bills, short in US, embodying the foreign exchange (FX) hedging component of the three-fund theorem of international finance (Solnik 1974). It assumes particular significance when the stock returns are expressed in home currency (here the NZD) as noted above. In that case the bond portfolio can be referred to as a currency hedge portfolio against the USD, and is in effect a forward contract. The relevant monthly return is defined by

$$r - (1 + r^*) \frac{e_1 - e_0}{e_1} - r^*$$

where:

$r$  = NZ one month bank bill rate as of start of month

$r^*$  = one month US CD rate as of start of month

$e$  = exchange rate as 1USD =  $e$  NZD, i.e. with the US as commodity currency and NZ as terms:  $e_1$  = end of period rate,  $e_0$  = beginning of period.

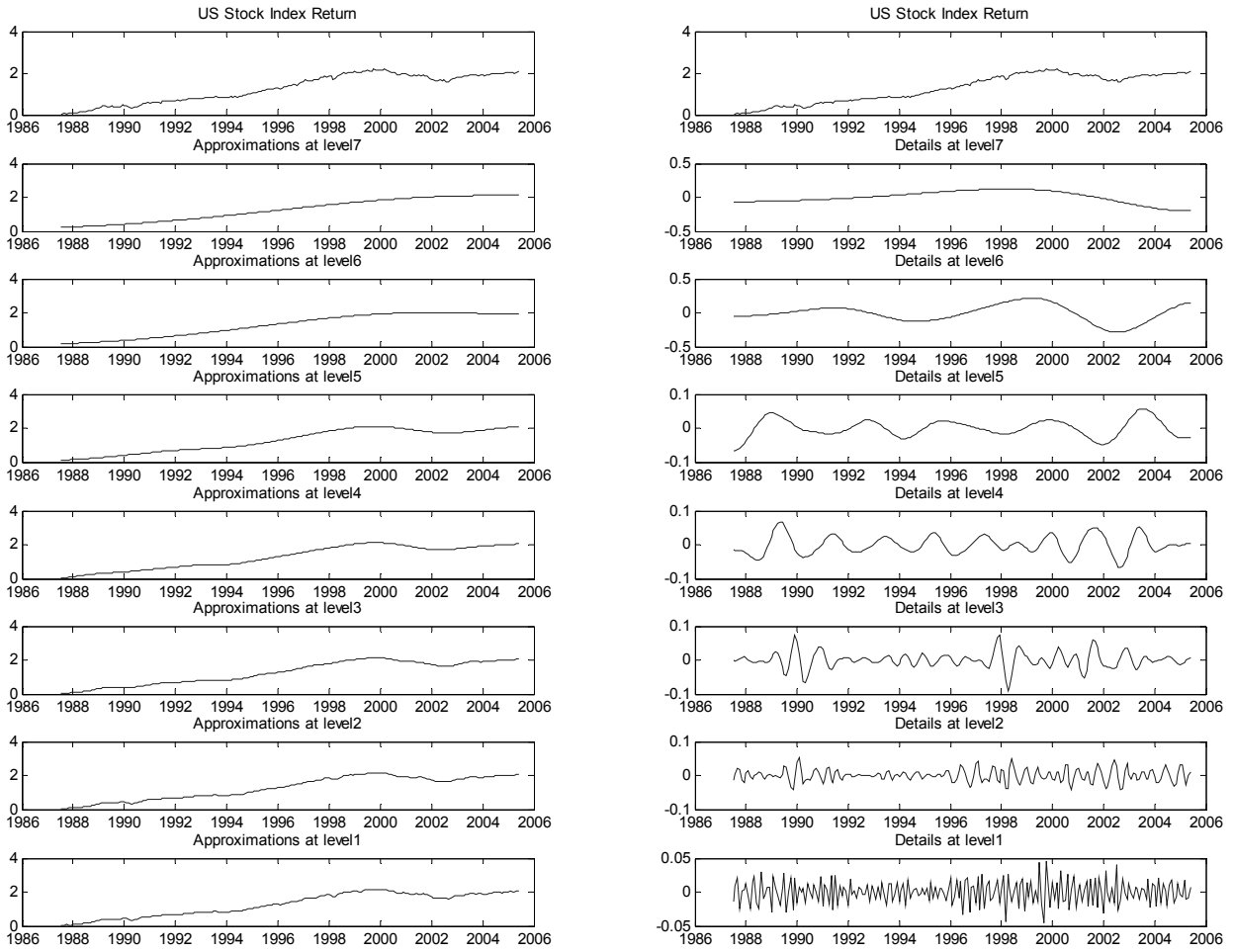


**Table 1: Average wavelet energy decomposition for asset classes**

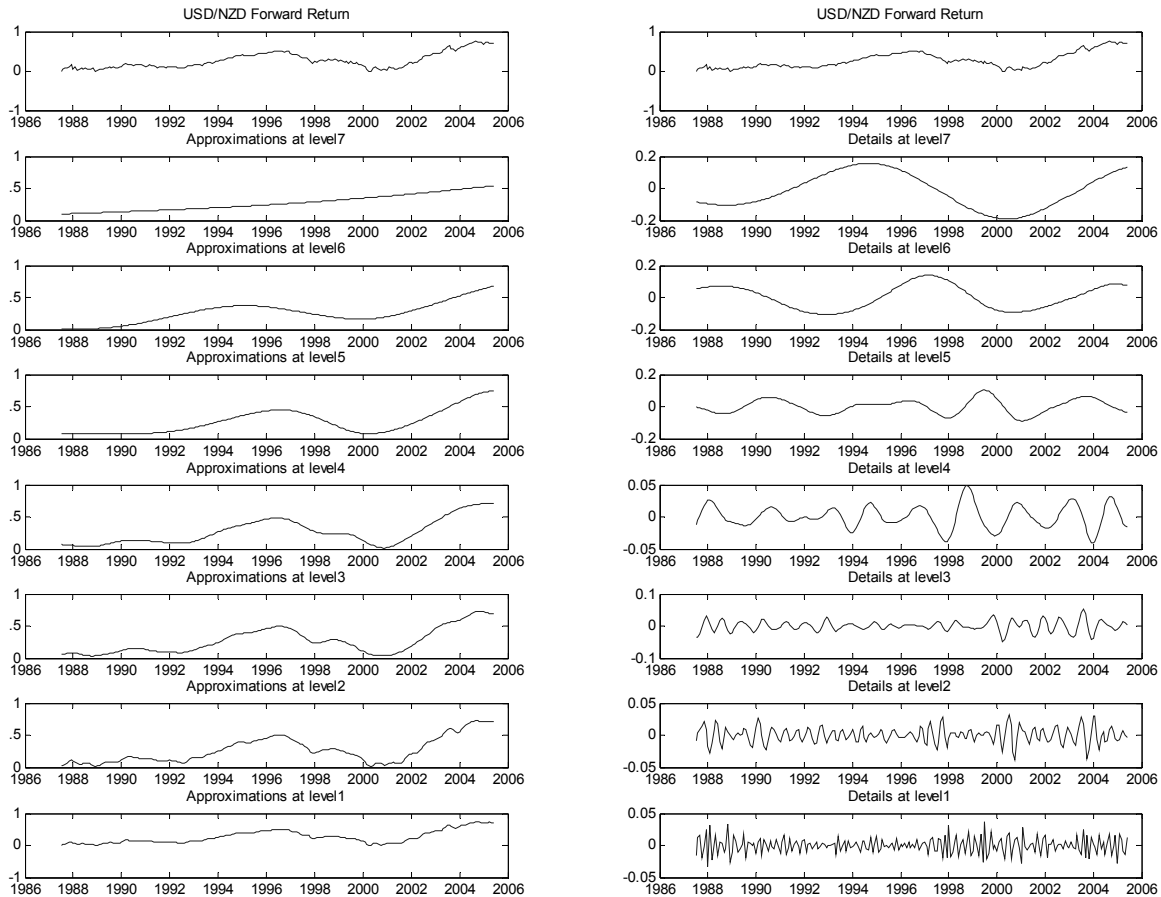
Detail level	Equivalent time period (months)	NZ Stocks Detail Energy (%)	US Stocks Detail Energy (%)	Japan Stocks Detail Energy (%)	Australia Stocks Detail Energy (%)	USD/NZD Forward Detail Energy (%)
1	2.9	2.41%	1.35%	2.15%	3.48%	1.00%
2	5.8	2.86%	1.58%	2.71%	4.02%	1.35%
3	11.6	4.84%	2.34%	5.26%	10.04%	2.85%
4	23.2	6.47%	3.38%	14.63%	16.06%	2.52%
5	46.4	5.57%	7.13%	22.46%	21.23%	5.95%
6	92.8	58.43%	59.30%	46.69%	30.87%	74.10%
7	185.6	19.42%	24.92%	6.11%	14.30%	12.23%

The power pattern in most cases shows an interior peak at level 6, which corresponds to an average cyclic period of 8.5 years. Note the higher energy of US stocks at longer run energies, especially detail 6.

The asset value cycles are irregular both as to amplitude and period. Figures 2a,b illustrate time series decompositions corresponding to figure 1 for the US stocks and the forward rate hedge instrument. It can be seen from these graphs that the details at level 7 and 6 have the highest power or energy among all the decomposed time series. However, the cycles are not necessarily regular as the details can have various amplitudes over time. By way of contrast, the standard exponential Ito model with constant drift has its power concentrated in level 7 and beyond, effectively becoming a stochastic trend. There is progressively little in detail levels below this, and certainly not an interior maximum as in table 1.



**Figure 2a: Wavelet decomposition of US stock index accumulated return**



**Figure 2b: Wavelet decomposition of USD/NZD forward rate hedge instrument accumulated return**

### III Structural components, dependent variables and path risk

#### *Choice of dependent variable*

As earlier noted, asset value increments often exhibit only weak short term dependence, the true power of macroeconomic effect becoming apparent only over longer intervals of time. Short-interval rates of return are dominated by market transients, which can include not only one-period white noise, but transients such as local bubbles or market overshooting in response to news or sentiment. Transient effects of this kind have been described by authors such as Barnett *et al* (1989), Lux (1995), Kojima (2000), Scheinkman and Xiong (2003). Market transients can generate episodic short term serial correlation that will magnify the variance of short term returns, but become smoothed out over longer horizons, allowing underlying fundamentals-based signals to become manifest. Thus wavelet analyses of the rates of return typically show energy increasing with detail – the higher the detail, the

greater the energy. The effect shows up in the present data (see Appendix 2). The situation is analogous to business cycle indicators, which are often prepared in two forms, the levels version and the growth or ‘growth cycle’ version, the latter referring to % rates of change (e.g. Bowden 2005 §5.4, for the US TCB indicators). Locating cycles and their turning points is easy with the first, but quite difficult with the second.

Our preference has been to use log asset values ( $\log V_t$ ) as an object, though there remains an issue as to whether this in turn should be normalised in some way.

One possible normalisation is to divide by the square root of time, as  $\frac{1}{\sqrt{t}} \log V_t$ .

To see why, suppose that returns are generated by the continuous time exponential process

$$dV_t = \mu V_t + \sigma V_t dB_t$$

where the cost of capital  $\mu$  just a constant and volatility  $\sigma$  is likewise constant, while  $B_t$  is a standard Brownian motion process. Taking  $V_0 = 1$  and dividing by  $\sqrt{t}$ , we can write the solution as

$$\frac{1}{\sqrt{t}} \log V_t = (\mu - 0.5\sigma^2)\sqrt{t} + \sigma \frac{B_t}{\sqrt{t}},$$

where  $B_t = \int_0^t dB_s$ . As  $\text{Var}(B_t) = t$ , it follows that  $\frac{1}{\sqrt{t}} \log V_t$  can be decomposed into

a trend element of order  $\sqrt{t}$  and a zero mean detail  $\sigma \frac{B_t}{\sqrt{t}}$ , which has constant variance

(energy, in this context). Thus one indication that the log value series is behaving like the classic accumulation process is that once normalised by  $\sqrt{t}$ , the details should show no obvious expansion in amplitude over time.

### *Component analyses*

On occasion, details might correspond to identifiable components. This is almost trivially true of the above exponential process, which can be written in the form

$$\log V_t = \log V_{1t} + \log V_{2t}, \text{ with}$$

$$V_{1t} = V_{10} e^{(\mu - 0.5\sigma^2)t}; \quad V_{2t} = V_{20} e^{\sigma B_t}.$$

The term  $V_{1t}$  can be thought of as the single detail encompassing the stochastic part, with  $V_{2t}$  as the residual approximation.

In general, however, seeking to identify components with details is not possible, simply because most assets are mutually related in one way or another and the orthogonal decompositions required by wavelet analysis will not apply. The closest analogy in Finance would be with factor models of returns, such as those used in the APT model (Ross 1976).

A more fruitful path is to first identify functional components on economic grounds, followed by a wavelet decomposition of each to test for mutual buffering, or on the other hand, amplification. This could be useful in studies of hedging or of other forms of portfolio enhancement. For instance, suppose a portfolio is divided into local ( $H$  for home) and foreign currency assets ( $F$ ), and let  $e$  be the exchange rate. So

$$V_t = V_{Ht} + e_t V_{Ft}$$

Then one can explore the advantage to adding foreign assets to a domestic base by working in terms of

$$\log\left(\frac{V_t}{V_{Ht}} - 1\right) = \log V_{Ft} + \log e_t - \log V_{Ht}$$

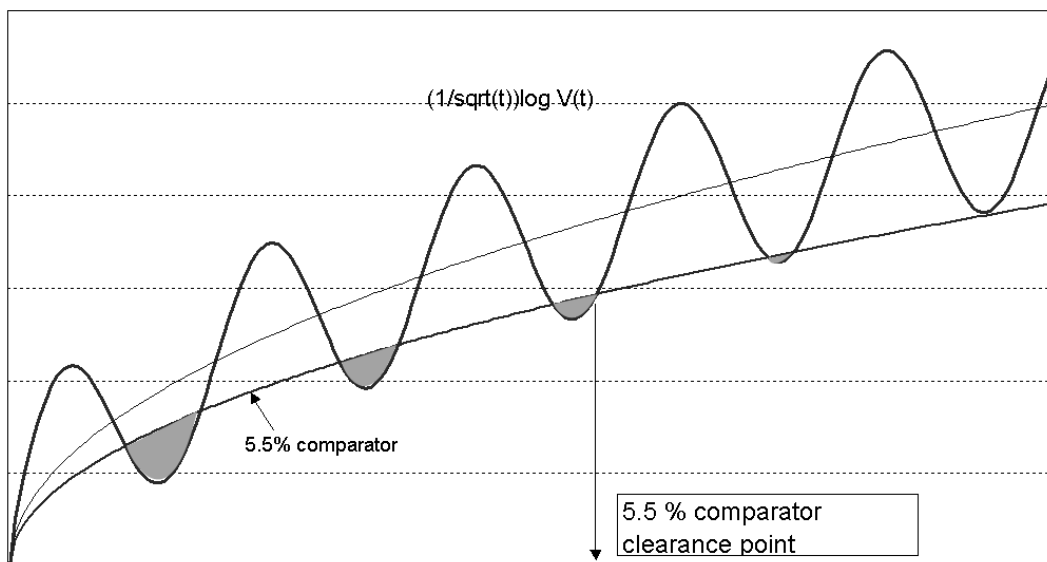
The right hand side can be regarded as the value to an enhancement portfolio (Bowden 2003) in which one has financed a foreign currency investment by going short the home portfolio. The enhancement portfolio value is now in a form amenable to a wavelet component analysis. One can ask questions such as whether the exchange rate buffers the foreign asset return or reinforces it and at what level of detail. A device essentially of this kind was used by Bowden and Zhu (2006), who showed that NZ farmers benefited until recently from a buffering relationship between world commodity prices and the NZ exchange rate, operating in the powerful 6-7 year wavelet detail.

#### *Comparator paths and dynamic value at risk*

Unforeseeable cyclical swings of large amplitude can create investor discomfort, especially if these occur at lower levels of detail, for it may take value longer to recover and in the meantime the investor will have become locked in. If this happens there will be investor regret that they that might have alternatively invested in some safer asset. They might even think that the depressed values were a signal of changed fund management or some other adverse contingency. Few investors would be unconcerned at a path that has an appreciable probability of values dipping below a

safe comparator, happening no matter that the original intent might have been to hold the investment for a very long time.

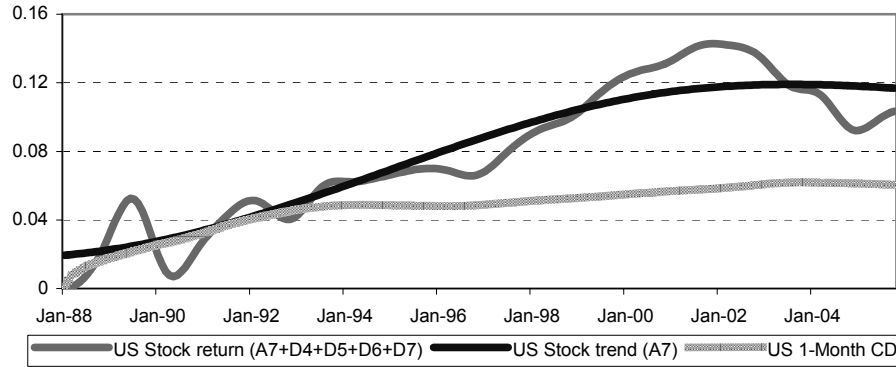
Figure 3 will serve to illustrate the resulting value at risk concept (if we suspend belief that the cycles as illustrated are irregular and unpredictable!). The investor could have chosen an asset with a safe annual return of 5.5% per annum (lower trend line). Investing in the fund would give a higher long-term expected growth path (upper trend line). But in the early years it would expose the investor to a chance that fund value would fall below the comparator and stay there for some time. Episodes where this has happened are indicated with the shaded areas. Only after some time has elapsed would the probability of this fall to below say 10%. The random stopping time where this happens can be called the 10% comparator value at risk clearance point. Long term cycles with high energy entail a longer stay in the value at risk 'sin bin' than do either shorter cycles, or cycles with lower energy.



**Figure 3: Dynamic VaR comparator clearance ( stylized)**

Figure 4 is a historical illustration, using the MSCI US total return index. The comparator path has been chosen as the one month US CD rate. Illustrated are both the trend on the stock index and the sum of details D4-D7, which carry the bulk of the cyclical power and are in themselves more threatening in terms of the lock-in effect. If we take the energy of the details as corresponding to the variance, the 10% VaR clearance point looks to be somewhere around 5 years from the start of the investment. After this date, there is only a progressively smaller probability of falling

below the comparator path. If the comparator path had been chosen as the long term bond rate this would have extended to about 10 years.



**Figure 4: Historical comparison of US stock index and 1-month CD return**

#### IV Portfolio choice

By combining assets into a portfolio one aims to achieve paths that are less risky for any given level of reward. In the familiar case of mean variance analysis, the reward is taken as the mean ( $\mu$ ) of returns. Risk is taken as the variance ( $\sigma^2$ ); we could call this the penalty element. To get the portfolio efficient frontier, one solves a programming problem that maximises the reward for a given level of risk, subject to constraints on the penalty element. Alternatively, one can formulate an expected utility function as

$$(1) \quad U = \mu - \lambda \sigma^2,$$

where  $\lambda$  is a trade-off parameter between reward and risk, and maximises  $U$  subject to feasibility constraints. The two approaches are almost equivalent. Thus the programming version, with  $\sigma^2$  to be less than or equal to some pre-assigned number, is locally equivalent to the expected utility version, with the parameter  $\lambda$  identifiable as the Lagrange multiplier at the optimal solution point. The equivalence between the two approaches has been fruitful in related contexts such as solving for portfolios in the presence of value at risk constraints (Bowden 2006). It can also be used in the present context. In this case, however, we do not have any natural notion of a mean, and the variance idea also differs to some degree. Thus the first task is to define appropriate reward and penalty elements.

In the wavelet based approach, we choose to carry out the portfolio analysis in terms of (log) values directly, and not returns as such. However, there is a natural relationship between the two. If a set of assets  $\{i\}$  have values  $V_i$ , then the asset

proportions are  $x_i = V_i / V$  and log accumulation per period is taken as  $\sum_i x_i \Delta \log V_i$ ,

which is a portfolio return. In some formulations, return elements can appear in the objective function (see below). However, in what follows wavelet approximations (A) and details (D) refer to log portfolio value.

*Reward or objective*

In long run investment decisions, the objective is most naturally taken as some metric assigned to the high level approximations, such that higher values at any time point are preferred to lower values. Let  $A_*$  denote the terminal approximation series i.e. that of maximal order consistent with historical data availability, taken as  $T$  observations. For brevity we shall sometimes loosely refer to this as the trend. In the preceding sections this was taken as  $A_7$ . A general objective might then be of the form

$$(2) \quad \text{Max } T \sum_{\tau=1}^T f(\tau) \Delta A_{*\tau},$$

which is a weighted sum of the historical value increments.

In expression (2),  $f(\tau)$  is a semipositive weighting function such that  $\sum_{\tau=1}^T f(\tau) = 1$ . It can be chosen to accord with some preference as between early or

late return accumulation. Useful special cases are as follows:

(i) The uniform weighting function  $f(\tau) = \frac{1}{T}$  all  $\tau$ , corresponds to the usual geometric rate of return over the whole horizon. Multiplying this by  $T$  as in the above objective gives  $A_{*T} - A_{*0}$ . Hence the objective is simply to maximise the terminal value of the trend. If the wavelet decomposition is carried out on logs to begin with, then the objective is the compounded value growth. If the factor  $T$  was missing in the objective (2), and logs were used, then the objective would be interpreted as the long run trend geometric average rate of return.

(ii) Take  $f(\tau) \propto \theta^{T-\tau}$ ;  $0 < \theta < 1$ . The higher the value chosen for  $\theta$ , the more we weight later values of value growth. The idea behind this is that later value increments of the historical record might have more predictive content for what is to come in the present real time.



*Penalties and constraints: band pass portfolios*

In passive or ‘strategic’ long-term investment plans, penalties are assigned to the detail energies, on the grounds that variation as such is regarded as unpleasant, exposing fund value to local losses before growth is eventually resumed.

The simplest way to handle differential welfare costs to energies is to combine them into a single constraint of the form

$$(3) \quad \sum_k w_k E_k \leq v; w_k \geq 0 \text{ and } \sum_k w_k = 1.$$

In expression (3),  $E_k$  denotes the average energy at detail level  $k$ ;  $\{w_k\}$  is a set of semipositive weights; and  $v$  is a user-assigned constant, interpreted as an average allowable energy. Thus by setting some of the  $\{w_k\}$  to zero, and assigning heavy penalties to the others, the resulting portfolios will favour variation in the former, but not the latter. One could call these band pass portfolios, motivated by similar usage in electronic system design, where one filters out signals at designated frequencies, allowing others to pass through unhindered. For example, a fund manager concerned that excessive short run fluctuations might unsettle investors, might elect  $w_k = 0$  for  $k > 2$ . This would allow lower level details to pass through unhindered while penalising short run fluctuations ( $k = 1, 2$ ).

The empirical illustration below takes an opposite tack, where the fund manager is driven by investor fear of large-scale opportunity losses. Suppose that the fund did experience a period of negative value growth. Investors would start actively considering whether to exit from the fund, especially if competitor funds were seen to be doing well. The alternative to exit (which itself involves penalties) is to wait until fund value recovers, if and when. But if the energy in longer details is high, the recovery may be years off instead of months. Investor outflow is more probable, leading to risks for fund sustainability and to future career prospects for fund managers. Conversely, if the bulk of energy is concentrated in shorter details, then the investor may feel better about simply staying put. Fund managers would have forewarned investors about short-term losses, so these are viewed with more equanimity. In effect, the economic holding cost is over short horizons. Alternative possibilities are mentioned below.

*Optimisation problem and equivalent utility function*

The optimisation problem is to choose the asset weights to maximise (2) subject to (3). Also incorporated are standard portfolio constraints, such as asset proportions

have to add up to unity if they require capital, or be semipositive if fund policy requires this. By varying the allowable energy parameter  $v$ , and solving the resulting portfolio, one can trace out an efficient frontier in just the same way as for classical mean-variance analysis.

The equivalent utility function is

$$(4) \quad U = T \sum_{\tau=1}^T f(\tau) \Delta A_{*,\tau} - \lambda \sum_k w_k E_k.$$

The utility function (4) represents a generalisation of the expected spectral utility function developed in Bowden (1977); see also Otrok (2001) for related constructs. Collectively the wavelet energies  $\{E_k\}$  correspond to the spectral power function  $f(w)$ , which are the variances of the demodulated series at each frequency. The weights  $\{w_k\}$  correspond to the spectral utility function  $U(w)$ . In the present contribution we have added a reward function, though this in itself has a mean dimension rather than a variance.

The parallel to the mean-variance utility function is also clear. Using the second mean value theorem, the effect of a weighted sum of energies is as though there is a single energy  $E_*$ , which in turn has the dimension of a variance. So the equivalence can be expressed as:

$$\sum_k w_k E_k = E_* \sim \sigma_*^2$$

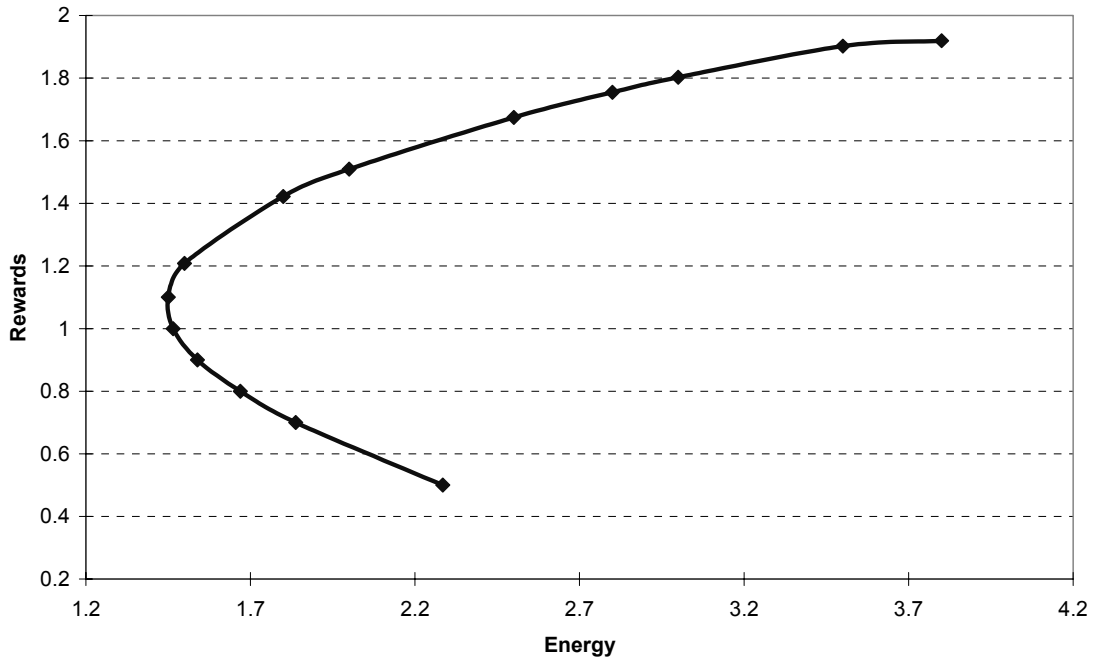
$$T \sum_{\tau=1}^T f(\tau) \Delta A_{*,\tau} \sim \mu.$$

The equivalence of the weighted sum of energies with a variance is useful in choosing the constant  $v$  in the programming specifications (see below).

### *Application*

Figure 5 depicts the reward - energy efficient frontier associated with the assets appearing in table 1. These are intended to be operational portfolios, so each of the stock returns have been converted to home currency. The portfolio devotes semi-positive weight to the five country stock market portfolios, and there is a single zero capital element, namely the USD/NZD forward contract, which can be shorted. In the objective function we used the time weight  $\theta = 0.9$  which has a mean distributed lag of 9 months, indicating that most of the value increment weight is assigned to the last 18 months of the horizon. For the energy weights  $\{w\}$  we assumed equal weights for

details 4-7 but zero weights for energy levels 1-3, i.e. that investors are unconcerned about short run fluctuations. In mean variance analysis, investors are assumed to be equally worried by short and long run power elements. Thus we are allowing power bands 1-3. i.e. the shorter run fluctuations, to pass freely.



**Figure 5: Efficient frontier**

The efficient frontier is strikingly similar to that of standard mean variance analysis, with the same parabolic shape extending into the lower inefficient half. As with mean variance, the trade-off (implied  $\lambda$  value) is higher as the energy bounds diminish.

Table 2 gives the optimal asset weights as one moves along the efficient frontier. Note that these do not have to add up to unity because of the presence of a zero capital element, namely the forward contract. Only the stock weights add up to unity. As the energy bounds become more restrictive, the optimal portfolio moves more to local NZ stocks, although the proportion devoted to US stocks is pretty much constant. It is also evident that the use of the USD/NZD forward contract diminishes. At first sight this looks puzzling; one might have expected a forward to be variance diminishing. However it is consistent with the finding of Thorp (2005) also Bowden and Zhu (2006), that preserving an exposure to unhedged spot USD/NZD is actually conservative risk management practice in Australasia.

It is of interest to see whether the similarity with mean-variance extends to the path properties of the optimal portfolios. If the two give similar results, this could be taken as comfort in the use of mean-variance analysis. If not, then the issue of optimality and long-term stability would have to be addressed, with the REF portfolio as a useful starting point.

Comparison of the two approaches can never be rigorous, as they refer to different reward or variation concepts, and standardisation is needed on one or the other. Our approach is to assume that locally dependent returns may be possible, so that wavelet analysis is the more appropriate. For the wavelet approach the specifications are chosen as above for the reward and energy weightings  $\theta$  and  $\{w_k\}$ . Two alternative comparisons along these lines are as follows.

(a) Normalise on the MV reward i.e. a given mean return over the entire horizon. Select the corresponding portfolio along the REF frontier that generates this mean. Compare the time paths and energy decompositions of the two portfolios, the one on the MV efficient frontier and the other on the REF frontier. Figure (6a) illustrates, while table (2a) gives the energies, and the respective portfolio compositions.

(b) Normalise on the wavelet energy  $E^*$ . Start with a MV efficient portfolio and calculate its total wavelet energy. Find the portfolio along the REF frontier that has the same total energy. Figure (6b) plots the two time paths of the resulting portfolios, while table (2b) contains the energies and portfolio compositions.

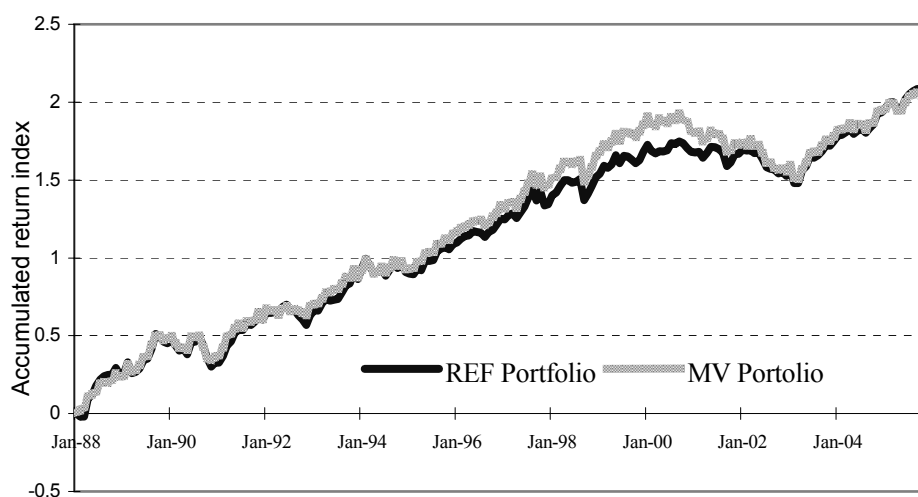
A third possible approach (not illustrated here) might be to normalise on the trade-off parameter  $\lambda$  between reward and variation, with appropriate interpretation of these dimensions in the respective contexts. Variation would be taken as  $\sigma^2$  for MV and as weighted energy for the REF portfolio.

Normalising on the reward as in (a) shows the smoothing effect of the wavelet based approach. The REF portfolio was slower to rise between 1996-2000, but with much less of a subsequent fall. The effect is apparent in the lower detail 6 energy. In portfolio terms, it is produced by down-weighting the US stocks component in favour of Australian stocks. The latter have virtually the same mean as the US, but materially lower long-term variation. The hedge proportion allocated to the US dollar has also diminished. The Japan weighting disappears altogether.

Normalisation (b), with fixed energy, results in a higher accumulation path for the REF portfolio. As before the Australian weight is increased at the expense of US stocks, but the tendency to use USD/NZD forwards remains roughly the same.

	<b>Mean-variance efficient portfolio MV</b>	<b>Reward-energy efficient portfolio REF</b>
<b>Assets</b>		
NZ	15.05%	19.57%
US	53.73%	31.26%
JP	10.13%	0.00%
AU	21.09%	49.17%
Forward	59.47%	39.43%
<b>Energies</b>		
E(D1)	0.04	0.05
E(D2)	0.06	0.07
E(D3)	0.11	0.15
E(D4)	0.18	0.25
E(D5)	0.43	0.44
E(D6)	1.67	1.16
E(D7)	0.26	0.15
<b>Mean</b>	0.00977	0.00977
<b>Variance</b>	0.001222	0.001408

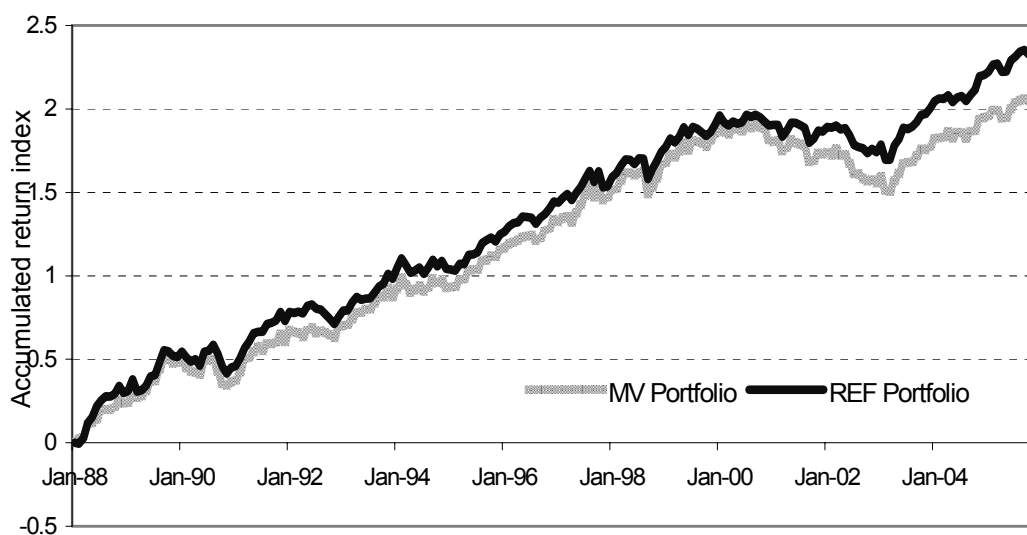
**Table 2a: Mean-normalised comparison between MV and REF portfolios**



**Figure 6a: Path comparison between MV and REF portfolios: mean-normalised**

	Mean-variance efficient portfolio MV	Reward- energy efficient portfolio REF
<b>Assets</b>		
NZ	15.05%	7.30%
US	53.73%	40.23%
JP	10.13%	0.00%
AU	21.09%	52.46%
Forward	59.47%	63.78%
<b>Energies</b>		
E(D1)	0.0448	0.0506
E(D2)	0.0555	0.0683
E(D3)	0.1086	0.1534
E(D4)	0.1825	0.2538
E(D5)	0.4343	0.4576
E(D6)	1.6674	1.5941
E(D7)	0.2575	0.2359
Mean	0.00977	0.01110
Variance	0.00122	0.00141

**Table 2b: Energy-normalised comparison between MV and REF portfolios**



**Figure 6b: Path comparison between MV and REF portfolios: energy-normalised**

### *Extensions*

Preceding development assumes that the manager's objective is to maximise long-term reward while minimising path risk. A quite different scenario might be that of a hedge fund concerned with identifying portfolios that actually maximise path risk over some designated detail band, perhaps one much shorter than assumed above. This could be accommodated by requiring a minimal reward element – or even deleting it – and maximising the energy  $E^*$  with an appropriate choice of the energy weights  $\{w_k\}$ . This looks a bit like a dual formulation from the classic theory of mathematical programming (Dantzig 1963, Rockafellar 1968, Murty 1976). However, the latter would require one to minimise the energy subject to reward constraints, so the 'hedge fund problem' is not quite dual to the long run strategic approach.

## **V Concluding remarks**

The underlying objective has been to develop portfolio technology that allows for non-independent return elements, dependence that may be weak in the short run but have a cumulative impact over the long run. There are other ways of attempting the same thing, notably by developing formal models of conditional returns, but they do require additional macroeconomic or time series modelling, which may be difficult. Explicit structural or time series modelling also requires the manager to vest a lot of faith in the predictive veracity of the model, which is problematic, given that economists are not all that good at long range forecasting of business cycles.

The wavelet based reward-energy approach is much less demanding in assumptions or informational requirements, than either mean-variance or formal causal modelling. On the other hand, it does have some maintained hypotheses of its own, notably that the long-term volatility patterns are characteristic of the data generation process, e.g. an underlying business cycle, and are likely to be repeated in the years to come. There is some comfort in the ability of wavelet analysis to detect structural breaks, which typically appear as sudden energy bursts in the high detail bands (Vuorenmaa 2005, also Bowden and Zhu 2006).

It may also be possible to endow the wavelet approach with *a priori* macroeconomic information, with the objective of increasing confidence in the long-term volatility patterns. For instance, if asset returns and value accumulation depend

on exchange rates, one might have a fair idea about the causal factors involved for a particular home country. In terms of our example, the NZ dollar is well known to be driven by world commodity price cycles, although the impact mechanism is itself variable. Cycles can therefore be expected. It is not possible to pick very precisely their timing and periodicity, but one can expect a considerable degree of power in the average 5-7 year detail zone.

Further points of contact in finance are with factor models of returns. In standard factor models such as the APT model, asset returns are generated in terms of unobservable orthogonal factors, but the latter continue to require the efficient market accumulation model. In the wavelet-based approach, the factors are orthogonal but no longer necessarily temporally independent. As with APT, there may exist dual portfolios that embody the factors. It would be of interest to explore what such portfolios might look like in different capital markets, and how they could be exploited in funds management or even as a basis for generic classes of fund.



## Appendix 1: Wavelet analysis

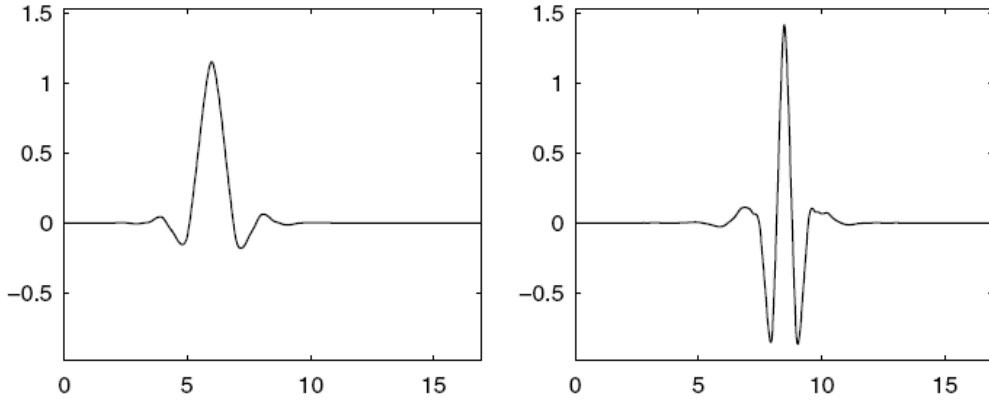
### Wavelet decompositions

Wavelets for a given family are generator functions, indexed by two parameters called the scale ( $j$ ) and the translation or location ( $k$ ). For the wavelet function we employed in the paper, namely coiflets, there are two different sorts of wavelets: the father wavelet  $\phi$  and the mother wavelet  $\psi$ . The former are normalised to integrate to unity, while the latter integrate to zero, as they are meant to span the cyclical influences. The two functions are respectively of the form:

$$\phi_{j,k}(t) = 2^{-j/2} \phi\left(\frac{t-2^j k}{2^j}\right); \quad \psi_{j,k}(t) = 2^{-j/2} \psi\left(\frac{t-2^j k}{2^j}\right).$$

The scale parameter determines the span of the wavelet, meaning its non-zero support, as each wavelet damps down to zero on either side of its centre. For a given time  $t$ , there are contributions from neighbouring wavelets translated to either side of  $t$ . The wavelet transform based on the above function is a dyadic procedure. Therefore, the maximum level decomposition of signal cannot exceed the integral part of  $\log_2^N$ , where  $N$  is the number of observations.

Figures 7a,b depicts the two wavelet generators used in the present study.



**Figure 7a: Coiflet father wavelet (left) and mother wavelet (right)**

The family of functions defined as above are mutually orthogonal. In a manner analogous to Fourier analysis one can form coefficients as

$$s_{J,k} = \int x(t) \phi_{J,k}(t) dt; \quad d_{j,k} = \int x(t) \psi_{j,k}(t) dt,$$

for  $j = 1, 2, \dots, J$ , where  $J$  is limited by the number of observations available on the given series  $x(t)$ , supposed continuous here for simplicity. As with the inverse transform in Fourier analysis, we can recover  $x(t)$  in terms the wavelet functions as:

$$x(t) = \sum_k s_{J,k} \phi_{J,k}(t) + \sum_k d_{J,k} \psi_{J,k}(t) + \sum_k d_{J-1,k} \psi_{J-1,k}(t) + \dots + \sum_k d_{1,k} \psi_{1,k}(t)$$

We write  $D_j(t) = \sum_k d_{j,k} \psi_{j,k}(t)$ . Note that just the one father wavelet has been used in the above, with maximal scale.

#### *Computational procedure*

The quasi Fourier approach illustrated above would be slow computationally. In the present paper, computations were done in Matlab (Misiti *et al* 2005) using Mallat's algorithm, which is considerably more efficient. The algorithm follows through the basic sequence as illustrated in figure 2 of the text. The original signal  $x(t)$  is fed through a high pass and low pass filter, one the quadrature of the other, which ensures orthogonality of the two outputs. The low pass filter is adapted to the longer run father wavelets and the higher to the mother wavelets. Output from the high pass filter is downloaded as the level 1 detail  $D_1$ , and the output from the low pass filter becomes the level 1 Approximation. Starting afresh with  $A_1$ , the process is successively repeated.

#### *Wavelet variances and covariances*

By decomposing the time series into orthogonal components as above, the variance of components at different scales can be derived. The DWT provides a simple way of computing these that closely parallels the classical statistical formulas. For each detail level  $j$ , the average energy or power over the horizon can be expressed as the percentage contribution of each level of detail relative to the whole as:

$$E_j^D = \frac{1}{E} \sum_t D_{j,t}^2, \quad E_j^A = \frac{1}{E} \sum_t A_{j,t}^2$$

$$E = \sum_t A_{j,t}^2 + \sum_j \sum_t D_{j,t}^2$$

The DWT variance computations can be improved using the maximal-overlap discrete wavelet transform (MODWT) estimator of the wavelet variance (Percival 1995). We have chosen not to use this as it assumes circularity, in other words the historical series simply repeats itself at each end.

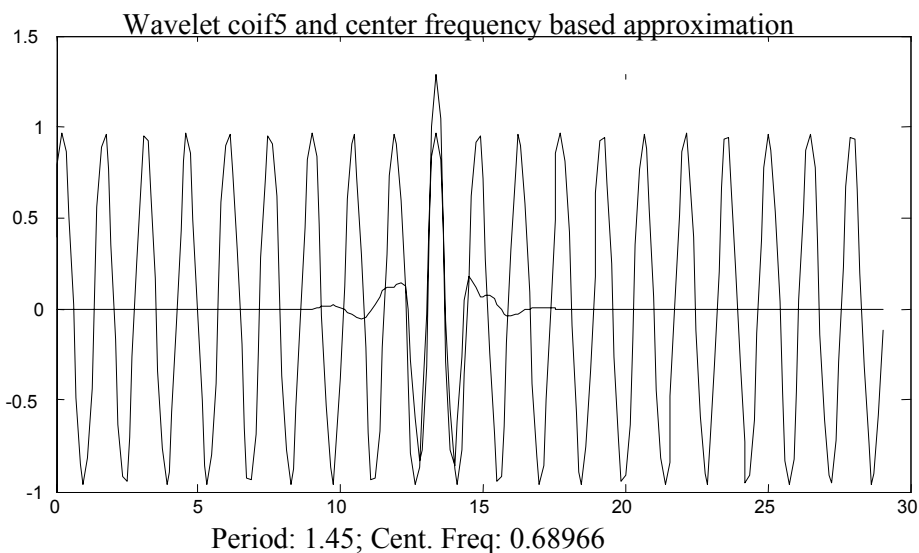
### Scale and frequency

To connect the scale to frequency, a pseudo frequency is calculated. The algorithm works by associating with the wavelet function a purely periodic signal of frequency  $F_c$  that maximizes the Fourier transform of the wavelet modulus. When the wavelet is dilated by the scaling factor  $2^j$ , the pseudo frequency corresponding to the scale is expressed as:

$$F_s = \frac{F_c}{2^j \times \Delta},$$

where  $\Delta$  is the sampling interval.

Taking the wavelet ‘coif5’ as an example, the centre frequency as seen from figure 10 is 0.68966 and thus the pseudo frequency corresponding to the scale  $2^5$  is 0.02155. As the sampling period is one month, the period corresponding to the pseudo frequency is 3.87 years.



**Figure 8: Scale in terms of equivalent sinusoidal frequency**

### Appendix 2: Energy in returns versus levels

Table 3 is an energy table for returns as distinct from the levels (values) used in the text. As the table indicates, wavelet energy now concentrates in the high detail band, and there is little indication of any interior maximum or other sign of power at higher details.

**Table 3: Energy decomposition for monthly asset returns**

	<b>NZ monthly return wavelet transform</b>	<b>US monthly return wavelet transform</b>	<b>JP monthly return wavelet transform</b>	<b>AU monthly return wavelet transform</b>	<b>USD/NZD forward monthly return wavelet transform</b>
<b>E(A7)(value)</b>	0.0054	0.0316	0.0069	0.0242	0.0013
<b>E(D1)</b>	0.3671	0.2267	0.4494	0.2476	0.0869
<b>E(D2)</b>	0.1456	0.1016	0.219	0.1025	0.0398
<b>E(D3)</b>	0.0885	0.0754	0.1077	0.0615	0.0243
<b>E(D4)</b>	0.0285	0.0248	0.062	0.0264	0.0077
<b>E(D5)</b>	0.0108	0.0086	0.0399	0.0126	0.0047
<b>E(D6)</b>	0.0035	0.0324	0.0229	0.0054	0.0103
<b>E(D7)</b>	0.0049	0.0037	0.0009	0.0006	0.0018

**Appendix 3: Data definitions and sources****Table 4: Data definitions and sources**

<b>Data</b>	<b>Definition</b>	<b>Resource</b>
MSCI USA	USA stock total return index	MSCI
MSCI NEW ZEALAND	New Zealand stock total return index	MSCI
MSCI AUSTRALIA	Australia stock total return index	MSCI
MSCI JAPAN	Japan stock total return index	MSCI
MSCI NZD TO 1 USD	Spot exchange rate (USD as the commodity currency and NZD as the terms currency, 1USD=SNZD)	MSCI
MSCI JPY TO 1 USD	Spot exchange rate (USD as the commodity currency and JPY as the term currency, 1USD=SJPY)	MSCI
AUSTRALIAN \$ TO US \$	Spot exchange rate (USD as the commodity currency and AUD as the term currency, 1USD=SAUD)	BB
NEW ZEALAND \$ TO US \$ 1MFWD	One month forward exchange rate (same expression as the spot rate)	BB
US CD 1M	US one month CD rate	US Federal Reserve Bank

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