Cashflow Immediacy and the Value of Investment Timing*

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Abstract

We examine the relationship between project value and cashflow immediacy when interest rates are uncertain and investment can be delayed. The value of investment delay has two components: the expected gain from committing now to investment at a future date and the potential gain from the ability to reverse this commitment. Holding the value of immediate investment constant, we show that the values of both components are increasing in the proportion of project cashflows that accrue in the more distant future, so total project value is greater for long-term projects. Our results emphasize the importance of the interaction between cashflow immediacy and interest rate uncertainty for the optimal investment policy.

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I Introduction

In perfect markets, standard investment theory holds that project value is given by the sum of discounted expected cashflows and is otherwise independent of project duration. However, Ingersoll and Ross (1992) show that the combination of interest rate volatility and flexibility in investment timing creates a role for duration. They consider a firm with a perpetual option to invest in a T-period project and find that the value of the rights to this project depends on T.

In this paper, we extend and elucidate this result. Ingersoll and Ross (1992) focus on a simple project that costs I dollars at the time of launching τ and yields a single \$1 payoff at time $\tau + T$. With this kind of project, any change in T also results in a change in net-present-value (NPV). Thus, the Ingersoll and Ross comparative static analysis of the relationship between T and project value does not hold all else constant.[1] Moreover, although the \$1 single-cashflow assumption facilitates comparison of projects with different T, it precludes analysis of the difference between multiple-cashflow projects that differ in the timing of cashflow arrival over a given project life.[2] To address these issues, we consider projects with multiple future cashflows, allow for changes in cashflow immediacy beyond those simply due to differences in project duration, and hold NPV constant when cashflow immediacy changes. We also incorporate more general assumptions about interest rates.

On the other hand, in order to make the economic intuition more transparent, we impose more restrictions on the investment timing structure than do Ingersoll and Ross (1992). They consider a

[1] Ingersoll and Ross consider two types of change in T. First, T is increased while I is held constant; clearly, this leads to a fall in NPV. Second, T is increased while the "breakeven" rate (the instantaneous spot rate for which NPV equals zero) is held constant; since the "breakeven" rate is not the project's internal rate of return, this change also results in a fall in NPV. See Narayanan (1985) for a discussion of the importance of keeping NPV constant when considering the effect of changes in project duration.

[2] For example, two projects can have the same NPV and duration, but one (the more "immediate" project) generates higher early cashflows and lower later cashflows than the other. In Sections IV and V of their paper, Ingersoll and Ross consider projects with multiple future cashflows, but do not analyze the effect of greater cashflow immediacy on the value of these more complex investments.

perpetual investment option whereas we permit the firm to invest at just two dates. This simpler structure allows us to identify and clarify the mechanism by which cashflow immediacy and interest rate uncertainty interact to affect project value while retaining the essential features of the timing decision.[3]

Although most readers are likely to be intuitively aware of most of our results, our analysis unifies these findings within a common framework and thus makes their underlying source more transparent. In the two-date set-up, project value is equal to the greater of (i) the payoff to immediate investment (the project's NPV) and (ii) the present value of the option to delay investment until the future date. Using this framework, we make two principal contributions. First, we show that it is useful to view the value of the option to delay investment as consisting of two components: the value of committing today to investment in the future and the value of being able to reverse this commitment if necessary. Because shocks to discount rates have a greater effect on the value of long-dated cashflows, and because investment payoffs are a convex function of discount rates, both components of the delay option value are greater for longer-term projects in the presence of interest rate uncertainty. Thus, for given NPV, project value is a decreasing function of cashflow immediacy. Second, we demonstrate the importance of assumptions about the term structure for the relationship between project value and cashflow immediacy. The assumptions implicit in Ingersoll and Ross (1992) yield the negative relationship described above, but more general interest rate environments that allow for twists in the yield curve can reverse this outcome. These results imply that dynamic investment decisions should consider the timing of project cashflows, particularly in a volatile interest rate environment.

In the next section, we outline the details of our model. In Section III, we use this model to examine the relationship between cashflow immediacy and project value in the presence of interest rate uncertainty. Section IV examines the importance of our assumptions about interest rate behavior while Section V offers some concluding remarks.

[3]The two-date assumption is, moreover, unimportant. In the appendix, we show that allowing the investment option to be perpetual does not change the sign of the relationship between cashflow immediacy and project value.

II The Model

Consider an investment project that incurs a sunk cost I and that lasts T years. At any launching time τ, $X(t)$ denotes the cumulative real cashflow expected t years after launching, that is at time $\tau + t$. We assume that the owners of this project (henceforth "the firm") are risk-neutral, so all expected cashflows are discounted at the applicable riskless interest rate. Although not strictly necessary, this assumption helps to keep our model simple and transparent.

At any time τ, interest rates and bond prices are a function of the n-vector of state variables $z_\tau \in \mathbb{R}^n$. Let $B(z_\tau, t)$ be the time τ price of a t-year riskless discount bond. Then time τ investment in the project yields the payoff[4]

$$\pi(z_\tau) = \int_0^T B(z_\tau, t)dX(t) - I. \tag{1}$$

We assume the firm has a simple investment timing choice: either invest now (at time 0) and receive an asset worth $\pi(z_0)$ or delay the investment decision until some future time s when the project is worth $\pi(z_s)$.[5] A necessary condition for current (time 0) investment in the project is $\pi(z_0) > 0$, so we henceforth assume this to be the case.

To determine the discount factors $B(z_\tau, t)$, let $f(z, t)$ denote the instantaneous t-year forward rate on a riskless discount bond when the state of the economy is z. Then (see Duffie, 1996, p150, or Yan, 2001)

$$B(z_\tau, t) = \exp\left(-\int_0^t f(z_\tau, t')dt'\right).$$

We assume that forward rates have the following properties.

(A1) For any value of z_s either $f(z_0, t) \geq f(z_s, t)$ for all $t \geq 0$ or $f(z_0, t) \leq f(z_s, t)$ for all $t \geq 0$.

(A2) $E_0[f(z_s, t)] \leq f(z_0, t)$ for all $t \geq 0$.

[4]Use of the Riemann-Stieltjes integral was suggested by the referee. The principal advantage of this approach is that it allows us to embed the simple Ingersoll and Ross (1992) project within our set-up: by setting $X(t) = 0$ for all $t < T$ and $X(T) = 1$, we obtain $\pi(z_\tau) = B(z_\tau, T) - I$, which is exactly the case considered by Ingersoll and Ross. On the other hand, if the project has a continuous cashflow $x(t)$, then $dX(t)$ can be replaced by $x(t)dt$ in the definition of π.

[5]This simple timing decision is similar to that analyzed by Abel et al. (1996)

Assumption (A1) asserts that shocks to the state vector move all instantaneous forward rates in the same direction while (A2) states that the distribution of the state vector is such that the instantaneous forward rate curve is expected to be no higher at time s than it is at time 0. We examine the importance of these assumptions in Section IV, but for the moment note only that both are implicit in Ingersoll and Ross (1992).

These assumptions allow for a wide variety of yield curves. To see this, note that the t-year yield-to-maturity at time s is

$$\frac{1}{t}\int_0^t f(z_s, t')dt'.$$

Thus, since we have placed no restrictions on the relationship between f and t, the yield curve can be monotonic or humped. Similarly, the yield-to-maturity of a long-term bond can be more or less sensitive to state vector shocks than the yield-to-maturity of a short-term bond, although these shocks move all yields in the same direction.

If the firm invests at time 0, it gives up the opportunity to invest at time s, so the cost of this sacrifice must be incorporated in the investment decision. From the perspective of time 0, the payoff from time s investment is random. At that time, if z_s is such that $\pi(z_s)$ is negative, the project is abandoned and the payoff is zero. For other values of z_s, investment at time s yields a positive payoff equal to $\pi(z_s)$. Thus, the future payoff from delaying investment is $\max\{0, \pi(z_s)\}$ and the value of the option to delay investment until time s is given by

$$\delta(z_0) = B(z_0, s)E_0[\max\{0, \pi(z_s)\}], \tag{2}$$

where the expectation is over all possible values of z_s. The time 0 value of the project is therefore given by

$$F(z_0) = \max\{\delta(z_0), \pi(z_0)\}$$

That is, project value equals the greater of (i) the value of immediate investment and (ii) the value of the option to delay investment until time s. In what follows, we hold the value of immediate investment $\pi(z_0)$ constant and examine the effect of changes in cashflow immediacy on the delay option value $\delta(z_0)$, and thus on project value $F(z_0)$.

To determine the nature of this relationship, it turns out to be helpful to decompose $\delta(z_0)$ into two parts — the value of cashflow delay, and the value of decision reversibility. To see how this decomposition works, imagine that the firm decides now to launch the project after a delay of s years, regardless of the state of the economy at that time. That is, it makes the investment decision now, but actually launches the project after a delay. Under such a policy, the current value of the project is

$$\delta_{CD}(z_0) = B(z_0, s)E_0[\pi(z_s)] \tag{3}$$

which we call the value of cashflow delay. However, we have already seen that the option to delay the investment decision until time s, which allows the firm to abandon the project if the economic climate is unfavorable, has current value $\delta(z_0)$. Using the identity $\max\{0, y\} - y = \max\{0, -y\}$ to subtract (3) from (2), we have

$$\delta_{DR}(z_0) = \delta(z_0) - \delta_{CD}(z_0) = B(z_0, s)E_0[\max\{0, -\pi(z_s)\}].$$

We can interpret this last term as the value of being able to reverse the investment decision. If $\pi(z_s) > 0$, there is nothing to be gained by reversing an earlier decision to invest; if $\pi(z_s) < 0$, reversing the earlier investment decision (i.e., abandoning the project) saves $-\pi(z_s)$.

The issue we wish to consider is whether or not project value depends on the timing of expected project cashflows beyond the extent to which this affects NPV. That is, does $F(z_0)$ depend on whether a project of given NPV has most of its expected cashflows in the near future or in the distant future? To address this question, it is convenient to consider a second project with greater cashflow immediacy than the first, but which is otherwise identical. If launched at time τ, this second project has expected cumulative cashflows $\hat{X}(t)$ at time $\tau + t$ such that $\hat{X}(t) - X(t)$ is an increasing function of t for all $t \in [0, t^*]$ and a decreasing function of t for all $t \in [t^*, T]$ for some time $t^* < T$; that is, the second project has higher early cashflows and lower later cashflows than the first project. Note that greater cashflow immediacy can indicate a shorter project duration, a re-weighting (towards the near future) of the expected cashflows over the same duration, or both. We use $\hat{\pi}(z_\tau)$ to denote the payoff from launching this more immediate project at time τ. Thus, holding NPV constant, $\hat{X}(t)$ also satisfies

$$\pi(z_0) = \int_0^T B(z_0, t)dX(t) - I = \int_0^T B(z_0, t)d\hat{X}(t) - I = \hat{\pi}(z_0),$$

Finally, we use $\hat{\delta}(z_0)$ to denote the time 0 value of the option to delay investment in the more immediate project until time s. The various components of this are defined in an analogous manner, i.e.,

$$\hat{\delta}(z_0) = \hat{\delta}_{CD}(z_0) + \hat{\delta}_{DR}(z_0)$$

and

$$\hat{F}(z_0) = \max\{\hat{\delta}(z_0), \hat{\pi}(z_0)\}.$$

Our analysis of the relationship between cashflow immediacy and project value is facilitated by the following technical result:

Lemma 1 *Let $u : [a, c] \to \mathbb{R}$ and $\hat{u} : [a, c] \to \mathbb{R}$ be increasing functions with the property that $\hat{u} - u$ is increasing on $[a, b]$ and decreasing on $[b, c]$, for some $b \in (a, c)$. For any continuous, increasing function $v : [a, c] \to \mathbb{R}$,*

$$\int_a^c v(t) d\hat{u}(t) - \int_a^c v(t) du(t) \leq v(b) \left(\int_a^c d\hat{u}(t) - \int_a^c du(t) \right).$$

Proof: See Appendix.

III Cashflow Immediacy and the Investment Decision

The first, and most important, point to note is that increasing cashflow immediacy makes the time s investment payoff less sensitive to interest rate shocks. To be precise, any change in the state vector which raises the payoff from investing in the more immediate project has an even greater positive effect on the investment payoff for the other project. Similarly, interest rate shocks which lower the investment payoff for the more immediate project have an even more negative effect on the other project. The intuition for this is straightforward — the more immediate the cashflows, the more they are concentrated in the short-term, where even large changes in the current state have a relatively small effect on their present value.

Lemma 2 *If (A1) holds, then the investment payoff functions π and $\hat{\pi}$ have the following properties:*

1. *If $\hat{\pi}(z_s) \geq \hat{\pi}(z_0)$ then $\pi(z_s) \geq \hat{\pi}(z_s)$.*

2. *If $\hat{\pi}(z_s) \leq \hat{\pi}(z_0)$ then $\pi(z_s) \leq \hat{\pi}(z_s)$.*

Proof: See Appendix.

Figure I: The effect of interest rate shocks on the investment payoff

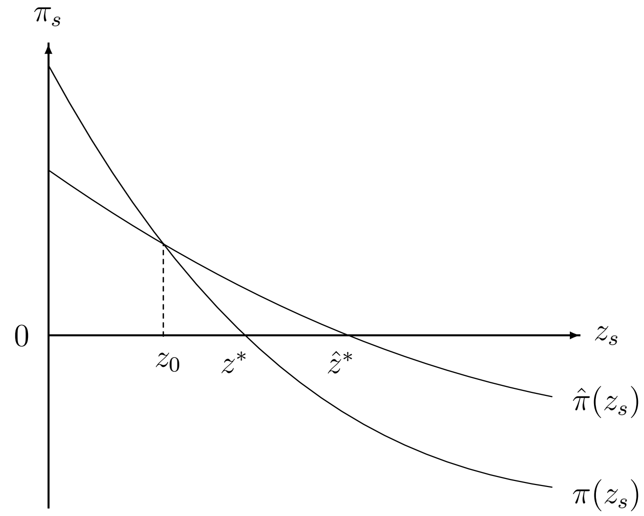

Figure I illustrates the relationship implied by Lemma 2 in the frequently-analyzed special case where the instantaneous spot interest rate is the sole state variable. By definition, the two projects have the same NPV, so they also have the same time s investment payoff if $z_s = z_0$. If the spot interest rate falls, the less immediate project has the greater payoff, since it is more sensitive to interest rate shocks. If the spot interest rate rises a small amount (to no more than z^*), both projects have positive payoffs, but the more immediate project has the greater payoff since it is affected less by the shock. If the spot interest rate rises to between z^* and $\hat{z}^*$, only the more immediate project has a positive payoff. For larger increases in the spot interest rate, neither project is launched and the payoffs are both zero. As any of these realizations are possible at time s, it is not immediately clear which project has the greater delay option value. However, decomposing this value into the two components introduced in Section II provides an unambiguous answer, as shown by the following two lemmas which specify the relationship between cashflow immediacy and the value of each component in the general case where z_s is an n-vector of state variables.

Lemma 3 *If (A1) holds, then the project with more immediate cashflows has a lower value of decision reversibility:*

$$\delta_{DR}(z_0) \geq \hat{\delta}_{DR}(z_0).$$

Proof: See Appendix.

Lemma 4 *If (A1) and (A2) hold, then the project with more immediate cashflows has a lower value of cashflow delay:*

$$\delta_{CD}(z_0) \geq \hat{\delta}_{CD}(z_0).$$

Proof: See Appendix.

Lemma 3 states that the value of decision reversibility is a decreasing function of cashflow immediacy. The option to reverse an earlier decision to commence investment at time s will only be exercised if new information reveals that decision to be a bad one. This will only occur if interest rates increase to such an extent that the investment payoff becomes negative. As can be seen in Figure I, there is a smaller likelihood of this happening when cashflows are more immediate: for the less immediate project, the investment decision will only be reversed if the state variable climbs above z^*; for the more immediate project, the state variable must climb above the higher threshold $\hat{z}^*$. Furthermore, when an investment decision is reversed, the savings are smaller when immediacy is high. This can also be seen in Figure I: when both payoffs are negative, the less immediate project's investment payoff is more negative.

Lemma 4 states that the value of cashflow delay is also a decreasing function of cashflow immediacy. With expected cashflows fixed, the time 0 expected payoff from a decision to invest at time s depends only on the distribution of the time s state variables. Since forward rates are expected to be no higher at time s, Jensen's inequality implies that $E_0[B(z_s, t)] > B(z_0, t)$. Thus, interest rate uncertainty increases the expected present value of cashflows generated by investment at time s. Moreover, this increase is greater for the less immediate project since it has more of its cashflows concentrated in the more distant future and long-term bond prices are more sensitive to interest rate movements than their short-term counterparts.

Lemmas 3 and 4 show that the value of each component of the option to delay investment is a decreasing function of cashflow immediacy. This yields our main result:

Proposition 1 *If (A1) and (A2) hold, then project value is a decreasing function of cashflow immediacy, all else equal.*

PROOF: From Lemmas 3 and 4,

$$\delta(z_0) = \delta_{DR}(z_0) + \delta_{CD}(z_0) \geq \hat{\delta}_{DR}(z_0) + \hat{\delta}_{CD}(z_0) = \hat{\delta}(z_0).$$

Therefore

$$F(z_0) = \max\{\delta(z_0), \pi(z_0)\} \geq \max\{\hat{\delta}(z_0), \hat{\pi}(z_0)\} = \hat{F}(z_0).$$

Proposition 1 states that an increase in cashflow immediacy lowers project value when interest rates are uncertain and NPV is held constant. The reason is that interest rate uncertainty creates the potential for a higher investment payoff at date s. This potential is greater the more "long-term" the project since discount rate fluctuations have a greater impact on more distant cashflows. Thus, the value of the option to delay investment becomes greater as cashflow immediacy decreases. Hence, when interest rates are uncertain, a long-term project has greater value than a short-term project with the same NPV. This contrasts with the Ingersoll and Ross (1992) finding that project value can be a decreasing function of project duration. This difference occurs because, in their model, a rise in duration is accompanied by a fall in NPV. This lowers project value and by more than greater interest rate sensitivity increases it. Our model makes it clear that when NPV is held constant, longer duration has an unambiguously positive effect on project value. However, as we discuss in the next section, this outcome is partly dependent on the term structure assumptions contained in both our model and that of Ingersoll and Ross.

IV Interest Rate Assumptions

The principal implication of our analysis is that, given assumptions (A1) and (A2), interest rate uncertainty and cashflow immediacy interact to affect project value in a systematic way. This begs the question of how this systematic relationship might be affected by the relaxation of the two assumptions and, more generally, of the exact nature of their role in our analysis.

Consider first assumption (A1) which requires that all instantaneous forward rates, and therefore bond yields, move in the same direction. This rules out "twists" in the yield curve that might reverse Lemma 2. For example, if shocks to the state vector move short-term and long-term yields in opposite directions, then long-term bond prices may be less sensitive to these shocks than short-term prices. In this case, the value of decision reversibility is higher for the more immediate project.

However, at least in single-factor models of the term structure, such "twists" are difficult to obtain; for example, in Cox, Ingersoll and Ross (1985) and Vasicek (1977), instantaneous forward rates are all increasing, affine functions of the single state variable (the instantaneous spot rate). In multi-factor models, things are not so clear. For example, suppose there are two sorts of shocks: one moves the short-term forward rate by more than the distant forward rate; the other results in parallel shifts of the forward curve. If the shocks hit in opposite directions, it is possible that short-term forward rates move in one direction, while distant forward rates move in the other direction. Thus, assumption (A1) appears likely to be satisfied in a single-factor world, but it may not hold in a multi- factor world, an outcome which can reverse Lemmas 2 and 3 and thus, if strong enough, could also reverse Proposition 1.

Assumption (A2) implies that the time s yield curve expected at date 0 is no higher than the curve prevailing at date 0. This restriction is implicit in the term structure models of Cox, Ingersoll and Ross (1980), Dothan (1978), and Ingersoll and Ross (1992), but not in the models of Cox, Ingersoll and Ross (1985) and Vasicek (1977) where a mean-reverting interest rate process can induce a higher expected future yield curve. In the latter case, project cashflows are expected to be more heavily discounted at date s and this effect is greater for less immediate projects because the discounted values of their cashflows are more sensitive to discount rate changes. Thus, an expected rise in interest rates reduces the size of the gap between $\delta_{CD}(z_0)$ and $\hat{\delta}_{CD}(z_0)$. If this effect is strong enough, the inequality in Lemma 4 reverses. However, note that Lemma 4 can accommodate a certain amount of expected yield curve increases. For example, because discount bond prices are a convex function of yields, Jensen's inequality implies that bond prices can have the martingale property even when the yield curve is expected to rise. In this case, the value of cashflow delay is independent of immediacy and the delay option value depends only on the value of decision reversibility. Thus, for Proposition 1 to be reversed, the expected growth in interest rates would have to be sufficiently high to offset the higher value of decision reversibility for the less immediate project. But expected interest rate growth lowers the incentive to delay investment, so any such reversal is likely to occur only when neither project has any significant delay option value. Indeed, this leads naturally to the suspicion that expected interest rate growth can cause the less immediate project to have a lower delay option value only in situations where both projects should be launched at date 0 and thus have equal value. Although we are unable to verify this conjecture for the two-date investment option, it is indeed valid for a perpetual option. We provide a brief outline of this latter case in the appendix.

V Concluding Remarks

In this paper, we have analyzed the effect of cashflow immediacy on project value when interest rates are uncertain and the timing of investment is flexible. We find that the option to delay investment is more valuable for less immediate projects, a result that has two sources. First, if the firm commits today to investment at some future date, then because investment payoffs are a convex function of discount rates, the current value of this commitment is greater for projects with less immediate cashflows. Second, the value of the ability to reverse this commitment is also higher for projects with less immediate cashflows; such projects are more sensitive to discount rate shocks, so the potential savings from being able to reverse the earlier commitment to invest are higher. Combining these two effects implies that long-term projects are more valuable than short-term projects with the same NPV. Thus, our results emphasize the importance of the interaction between cashflow immediacy and interest rate uncertainty for the optimal investment policy.

Our analysis has been conducted in a setting where delaying investment does not affect expected cashflows and thus has particular relevance to decisions involving investments of low cashflow risk and varying duration.[6] This raises the question of whether allowing expected cashflows to vary if investment is delayed can induce similar results. To see how this might occur, recall the mechanism that operates when interest rates are stochastic and expected cashflows are constant. In that case, project value is a decreasing function of cashflow immediacy because shocks to interest rates have a proportionately greater effect on the discount factors applied to more distant expected cashflows. Thus, from the perspective of time 0, the potential gains from delaying investment until time s are greater for projects with a relatively high proportion of more distant cashflows. Since the present value of any expected cashflow is simply equal to the product of that expected cashflow and the relevant discount factor, this suggests that greater cashflow immediacy should reduce the potential

[6]As pointed out to us by the referee, banks frequently face decisions of this type.

gains from delaying investment even when interest rates are certain, so long as time s shocks to the project's expected cashflow stream have a proportionately greater effect on the more distant expected cashflows. Suppose, for example, that the expected cashflow state variable is the realized growth rate in the price of the good which the project will produce and that there is some persistence in price growth realizations. Then a positive shock increases all expected future cashflows, but the percentage increase is greatest for later cashflows because of the persistence effect. This suggests that cashflow immediacy may have implications for project value even when interest rate risk is low.

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Appendix

Proof of Lemma 1

Since $v(t) \le v(b)$ for all $t \in [a, b]$,

$$\int_a^b v(t)d(\hat{u}(t) - u(t)) \le v(b) \int_a^b d(\hat{u}(t) - u(t)) = v(b) \int_a^b d\hat{u}(t) - v(b) \int_a^b du(t).$$

Since $v(t) \ge v(b)$ for all $t \in [b, c]$,

$$\int_b^c v(t)d(u(t) - \hat{u}(t)) \ge v(b) \int_b^c d(u(t) - \hat{u}(t)) = v(b) \int_b^c du(t) - v(b) \int_b^c d\hat{u}(t).$$

The result of the lemma follows immediately.

Proof of Lemma 2

Let $a = 0$, $b = t^*$, $c = T$, $u(t) = \int_0^t B(z_0, t')dX(t')$ and $\hat{u}(t) = \int_0^t B(z_0, t')d\hat{X}(t')$.

First consider the case that $f(z_s, t) \le f(z_0, t)$ for all t. Then clearly $B(z_s, t) \ge B(z_0, t)$ for all t, which implies that $\hat{\pi}(z_s) \ge \hat{\pi}(z_0)$. Furthermore,

$$v(t) = \frac{B(z_s, t)}{B(z_0, t)} = \exp\left(\int_0^t (f(z_0, t') - f(z_s, t'))dt\right)$$

is an increasing function of t. We see from Lemma 1 that

$$\begin{aligned}
\hat{\pi}(z_s) - \pi(z_s) &= \int_0^T B(z_s, t)d\hat{X}(t) - \int_0^T B(z_s, t)dX(t) \\
&= \int_0^T v(t)d\hat{u}(t) - \int_0^T v(t)du(t) \\
&\le v(t^*)\left(\int_0^T d\hat{u}(t) - \int_0^T du(t)\right) \\
&= \frac{B(z_s, t^*)}{B(z_0, t^*)}(\hat{\pi}(z_0) - \pi(z_0)) \\
&= 0,
\end{aligned}$$

where we have used the fact that $\pi(z_0) = \hat{\pi}(z_0)$ in the final line.

The only other possibility is that $f(z_s, t) \ge f(z_0, t)$ for all t. Then clearly $B(z_s, t) \le B(z_0, t)$ for all t, which implies that $\hat{\pi}(z_s) \le \hat{\pi}(z_0)$. Furthermore,

$$v(t) = \frac{-B(z_s, t)}{B(z_0, t)} = -\exp\left(\int_0^t (f(z_0, t') - f(z_s, t'))dt\right)$$

is therefore an increasing function of t. We see from Lemma 1 that

$$\begin{aligned} -\hat{\pi}(z_s)+\pi(z_s) &= -\int_0^T B(z_s,t)d\hat{X}(t)+\int_0^T B(z_s,t)dX(t) \\ &= \int_0^T v(t)d\hat{u}(t)-\int_0^T v(t)du(t) \\ &\leq v(t^*)\left(\int_0^T d\hat{u}(t)-\int_0^T du(t)\right) \\ &= \frac{-B(z_s,t^*)}{B(z_0,t^*)}(\hat{\pi}(z_0)-\pi(z_0)) \\ &= 0. \end{aligned}$$

Proof of Lemma 3

Let $A=\{z\in\mathbb{R}^n:\pi(z)\leq 0\}$ and $\hat{A}=\{z\in\mathbb{R}^n:\hat{\pi}(z)\leq 0\}$. For any $z\in\hat{A}$, $\hat{\pi}(z)\leq\hat{\pi}(z_0)$, so that Lemma 2 implies $\pi(z)\leq\hat{\pi}(z)\leq 0$, and $z\in A$. That is, $\hat{A}\subseteq A$, and hence

$$\iint_A g(z)(-\pi(z))dz \geq \iint_{\hat{A}} g(z)(-\pi(z))dz,$$

where g is the probability density function for z_s. Also, we must have $-\pi(z)\geq-\hat{\pi}(z)$ for all $z\in\hat{A}$. Therefore

$$\iint_{\hat{A}} g(z)(-\pi(z))dz \geq \iint_{\hat{A}} g(z)(-\hat{\pi}(z))dz.$$

Combining these two results, we see that

$$\begin{aligned} E_0[\max\{0,-\pi(z_s)\}] &= \iint_A g(z)(-\pi(z))dz \\ &\geq \iint_{\hat{A}} g(z)(-\hat{\pi}(z))dz \\ &= E_0[\max\{0,-\hat{\pi}(z_s)\}]. \end{aligned}$$

Proof of Lemma 4

Our first task is to show that the function $E_0[B(z_s,t)/B(z_0,t)]$ is increasing in t. To begin, note that

$$\frac{d}{dt}\left(E_0\left[\frac{B(z_s,t)}{B(z_0,t)}\right]\right) = E_0\left[\frac{B(z_s,t)}{B(z_0,t)}\left(\frac{\frac{\partial}{\partial t}B(z_s,t)}{B(z_s,t)}-\frac{\frac{\partial}{\partial t}B(z_0,t)}{B(z_0,t)}\right)\right]$$

$$
\begin{aligned}
&= E_0\left[\frac{B(z_s,t)}{B(z_0,t)}(f(z_0,t)-f(z_s,t))\right] \\
&= \iint_{\mathsf{R}^n} g(z)\frac{B(z,t)}{B(z_0,t)}(f(z_0,t)-f(z,t))dz.
\end{aligned}
$$

Now let $B = \{z \in \mathbb{R}^n : f(z_0,t) \geq f(z,t)\ \forall t \geq 0\}$ and $C = \{z \in \mathbb{R}^n : f(z_0,t) \leq f(z,t)\ \forall t \geq 0\}$. Then

$$
\begin{aligned}
\frac{d}{dt}\left(E_0\left[\frac{B(z_s,t)}{B(z_0,t)}\right]\right) &= \iint_B g(z)\frac{B(z,t)}{B(z_0,t)}(f(z_0,t)-f(z,t))dz \\
&\quad + \iint_C g(z)\frac{B(z,t)}{B(z_0,t)}(f(z_0,t)-f(z,t))dz
\end{aligned} \tag{4}
$$

For all $z \in B$, $B(z,t)/B(z_0,t) \geq 1$ for all $t > 0$. Therefore

$$
\iint_B g(z)\frac{B(z,t)}{B(z_0,t)}(f(z_0,t)-f(z,t))dz \geq \iint_B g(z)(f(z_0,t)-f(z,t))dz.
$$

Similarly, for all $z \in B$, $B(z,t)/B(z_0,t) \leq 1$ for all $t > 0$, so that

$$
\iint_C g(z)\frac{B(z,t)}{B(z_0,t)}(f(z_0,t)-f(z,t))dz \geq \iint_C g(z)(f(z_0,t)-f(z,t))dz.
$$

Substituting these two inequalities into (4) shows that

$$
\begin{aligned}
\frac{d}{dt}\left(E_0\left[\frac{B(z_s,t)}{B(z_0,t)}\right]\right) &\geq \iint_B g(z)(f(z_0,t)-f(z,t))dz \\
&\quad + \iint_C g(z)(f(z_0,t)-f(z,t))dz \\
&= \iint_{\mathsf{R}^n} g(z)(f(z_0,t)-f(z,t))dz \\
&= E_0[f(z_0,t)-f(z_s,t)] \\
&= f(z_0,t) - E_0[f(z_s,t)] \\
&\geq 0,
\end{aligned}
$$

where we have used (A2) to complete the final step.

The remainder of the proof is easy. By setting $a = 0$, $b = t^*$, $c = T$, $u(t) = \int_0^t B(z_0,t')dX(t')$, $\hat{u}(t) = \int_0^t B(z_0,t')d\hat{X}(t')$ and $v(t) = E_0[B(z_s,t)/B(z_0,t)]$, we see from Lemma 1 that

$$
\begin{aligned}
E_0[\hat{\pi}(z_s)] - E_0[\pi(z_s)] &= \int_0^T v(t)d\hat{u}(t) - \int_0^T v(t)du(t) \\
&\leq v(t^*)\left(\int_0^T d\hat{u}(t) - \int_0^T du(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= E_0\left[\frac{B(z_s,t^*)}{B(z_0,t^*)}\right](\hat{\pi}(z_0)-\pi(z_0)) \\
&= 0.
\end{aligned}
$$

Perpetual Options

We consider the effects of allowing (i) the investment option to be perpetual and (ii) positive expected interest rate growth. Let $R(r_t)$ denote the value of perpetual one-time rights to the project introduced in Section II where the state variable r_t is the instantaneous spot interest rate. We suppose that r_t follows the (risk-adjusted) process

$$dr_t = \mu(r_t)dt + \sigma(r_t)d\xi_t$$

for some functions μ and σ, where ξ_t is a Wiener process.

Denote the price at time τ of a discount bond paying \$1 at time $\tau + t$ by $B(r_\tau, t)$.[7] As before, investing in the project at time τ yields the payoff given in equation (1):

$$\pi(r_\tau) = \int_0^T B(r_\tau, t)dX(t) - I.$$

Assuming that the project is launched as soon as the instantaneous rate takes the value r^*, R must satisfy the ordinary differential equation

$$0 = \frac{1}{2}\sigma(r)^2\frac{d^2R}{dr^2} + \mu(r)\frac{dR}{dr} - rR, \quad r > r^*,$$

together with the boundary conditions

$$\lim_{r\to\infty} R(r) = 0, \quad R(r^*) = \pi(r^*).$$

The first condition says that the rights to the project have zero value when the instantaneous rate is extremely high, the second condition reflects the fact that the investment payoff is $\pi(r^*)$. Following the approach in Section IV of Ingersoll and Ross (1992), we can write

$$R(r) = \frac{\phi(r)\pi(r^*)}{\phi(r^*)},$$

[7]The function B can be found by solving a partial differential equation analagous to equation (5) in Ingersoll and Ross.

where the function ϕ satisfies the ordinary differential equation

$$0 = \frac{1}{2}\sigma(r)^2 \frac{d^2\phi}{dr^2} + \mu(r)\frac{d\phi}{dr} - r\phi,$$

together with the boundary condition $\lim_{r\to\infty} \phi(r) = 0$.[8] The optimal investment threshold is the interest rate r' which maximizes $\pi(r')/\phi(r')$, and the value of the rights to the project is therefore

$$R(r) = \sup_{r'} \frac{\phi(r)\pi(r')}{\phi(r')} = \phi(r) \sup_{r'} \frac{\pi(r')}{\phi(r')}.$$

Now consider the more immediate project with investment payoff function $\hat{\pi}(r)$. If $\hat{r}^*$ is the optimal threshold for the project with more immediate cashflows, the value of the rights to this project is

$$\hat{R}(r_0) = \frac{\phi(r_0)\hat{\pi}(\hat{r}^*)}{\phi(\hat{r}^*)}.$$

Provided that all forward rates are increasing functions of the instantaneous spot rate, (A1) holds, and Lemma 2 implies that

$$\pi(r) \geq \hat{\pi}(r) \quad \forall\, r \leq r_0.$$

If it is initially optimal to delay investment on the second project (that is, $\hat{r}^* \leq r_0$), then Lemma 2 implies that $\pi(\hat{r}^*) \geq \hat{\pi}(\hat{r}^*)$. Therefore,

$$\frac{\phi(r_0)\hat{\pi}(\hat{r}^*)}{\phi(\hat{r}^*)} \leq \frac{\phi(r_0)\pi(\hat{r}^*)}{\phi(\hat{r}^*)}.$$

Finally, $\hat{r}^*$ is not necessarily the optimal threshold for the project with payoff function π, so that

$$\frac{\phi(r_0)\pi(\hat{r}^*)}{\phi(\hat{r}^*)}. \leq \phi(r_0) \sup_{r'} \frac{\pi(r')}{\phi(r')} = R(r_0).$$

Combining the last three results shows that

$$\hat{R}(r_0) \leq R(r_0).$$

So long as any expected increase in r (and therefore in yields) is not too great, the strict inequality holds, i.e., the rights to the more immediate project are less valuable. For large expected increases in r, delay is not optimal and the two projects are equivalent.

[8]Examples of ϕs for popular models from the term structure literature are given in Table 5 of Ingersoll and Ross (1992).