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Electrostatics of a family of conducting toroids

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Abstract

An exact solution is found for the electrostatic potential of a family of conducting charged toroids. The toroids are characterized by two lengths a and b, with $a \geqslant 2b$. They are closed, with no hole in the 'doughnut'. The results are obtained by considering the potential of two equal charges, displaced from the origin by equal and opposite imaginary translations. All the electrostatic properties, including the surface charge density, are expressed in terms of elementary functions, and the topic is thus suitable for advanced class projects. A set of suggested class problems is included.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

An isolated conductor at potential V_0 and carrying a charge Q has capacitance $C = Q/V_0$. A sphere of radius a has $C = a$ in Gaussian units (in SI units, $C = 4\pi\varepsilon_0 a$) and a thin disk of radius a has $C = \left(\frac{2}{\pi}\right)a$. A prolate ellipsoid of revolution, with semi-major axis a and semi-minor axis b, has [1]

$$C = \frac{\sqrt{a^2 - b^2}}{\ln\left[\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right]}. \tag{1}$$

(The capacitance of a sphere is contained within (1) as the limit $b \to a-$, and (1) is in turn contained in an incomplete elliptic integral for the capacitance of a general ellipsoid [2].) This short list seems to exhaust the known closed-form results for the capacitance of isolated conductors. There are known bounds for the capacitance of the regular solids [3], and numerical bounds for the square and the cube [4].

For the circular (anchor ring) torus, there is an exact infinite series expression for the capacitance [5] in terms of toroidal harmonics, a perturbation method [6], an integral equation approach [7] and calculation of electrostatic excitation [8]. In the case of a tight circular torus (hole just closed), the capacitance can be expressed as an infinite integral over modified Bessel functions [9].

Here we shall give closed-form expressions for the equipotentials of a set of toroidal surfaces which are characterized by two positive lengths a and b, with $a \geqslant 2b$. (We shall use the term 'toroidal surface' as a descriptor of this family, even though the electrostatically interesting surfaces do not have a hole in the middle. This is because the family includes surfaces which look like conventional tori, namely those with $a < 2b$, and also because the ring $\rho = b$, $z = 0$ is pivotal to the geometry of the whole family.)

The $a \geqslant 2b$ tori are closed, like spheres for $a \gg b$, flattened at the poles for $a \geqslant (5/2)b$. There is a conical indentation at the poles when $a = 2b$. Figure 1 illustrates the shapes when $a = 2b$ and $3b$. In all cases, the capacitance of the conducting torus is equal to a.

2. Equipotentials of two equal charges with imaginary displacements

The reader will be familiar with equipotentials resulting from two charges (equal or not) at some distance from each other. An example may be seen in figure 1.08b of [5]. Here we consider imaginary displacements of the charges along the symmetry axis, $\Delta z = \pm \mathrm{i}b$, so the complex distances to a field point at (ρ, z) in cylindrical coordinates ($\rho = x^2 + y^2$) from charges at $(0, \mp \mathrm{i}b)$ are

$$r_\pm^2 = \rho^2 + (z \pm \mathrm{i}b)^2 = \rho^2 - b^2 + z^2 \pm 2\mathrm{i}bz. \tag{2}$$

Let each charge be $Q/2$, so the potential is given by

$$V(\rho, z) = \frac{Q}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right). \tag{3}$$

This is a real potential function, and as the Laplacian is unchanged by a translation in z, real or complex, it satisfies $\nabla^2 V = 0$. What are the shapes of the equipotential surfaces?

Since both r_+ and r_- are zero on the circle $\rho = b$ in the $z = 0$ plane, we might guess that (3) represents the potential of a charged ring. Not so: the charged ring potential is given by an elliptic integral of the first kind [10, 11], and has a logarithmic singularity on the critical circle $\{\rho = b, z = 0\}$. In contrast, the equipotentials of (3), defined by

$$V(\rho, z) = Q/a \qquad \text{or} \qquad a(r_+ + r_-) = 2r_+r_-, \tag{4}$$

are algebraic, and the singularity on the critical circle is inside the smallest ($a = 2b$) physical equipotential surface.

Elimination of the square roots from (4) by two squarings leads to an eighth-degree equation linking ρ and z with the critical ring parameter b and the equipotential parameter a:

$$\begin{aligned} Z = \rho^8 &+ (4z^2 - 4b^2 - a^2)\rho^6 + [6z^4 - z^2(3a^2 + 4b^2) + 3b^2(a^2 + 2b^2)]\rho^4 \\ &+ [4z^6 - (3a^2 - 4b^2)z^4 + 2b^2(a^2 - 2b^2)z^2 - b^4(3a^2 + 4b^2)]\rho^2 \\ &+ z^8 + (4b^2 - a^2)z^6 + b^2(6b^2 - a^2)z^4 \\ &+ b^2(4b^4 + a^2b^2 - a^4)z^2 + b^6(a^2 + b^2) = 0. \end{aligned} \tag{5}$$

When $b \to 0$, the constraint (5) reduces to a sphere of radius a:

$$Z \to (\rho^2 + z^2)^3(\rho^2 + z^2 - a^2). \tag{6}$$

(There is then a charge Q at the origin, with spherical equipotential surfaces.) When $a \to 0$ ($V \to \infty$), Z shrinks to the critical circle:

$$Z \to [(\rho - b)^2 + z^2]^2[(\rho + b)^2 + z^2]^2. \tag{7}$$

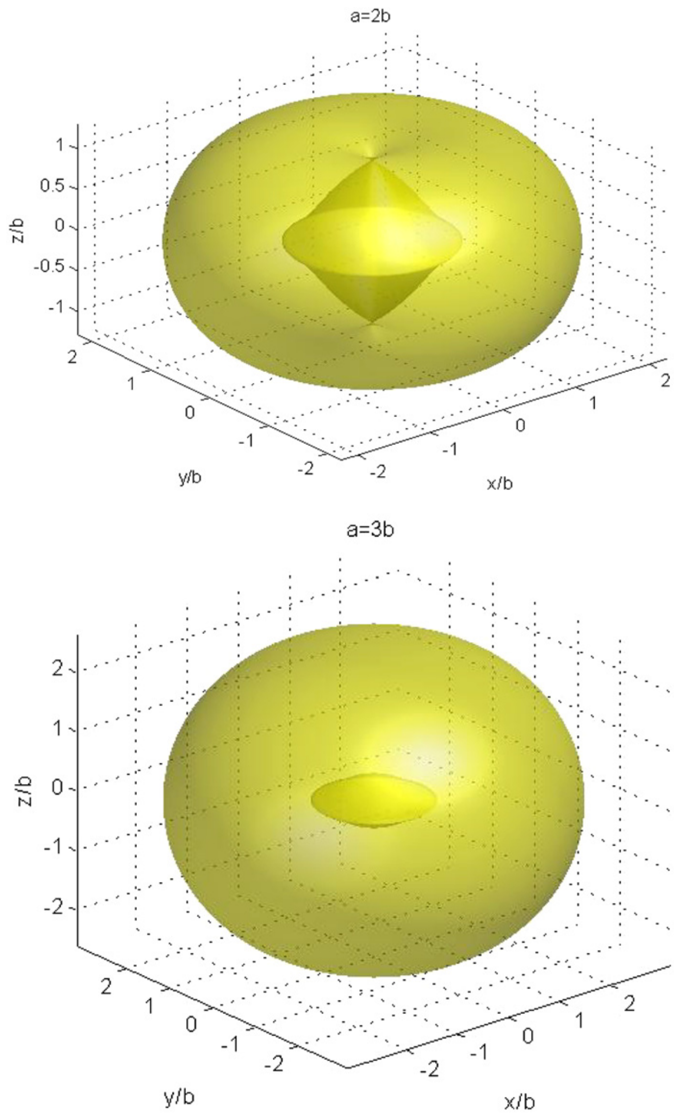

Figure 1. Upper figure: the equipotential toroidal surface for $a = 2b$. Note the conical indentations at the poles. Lower figure: the $a = 3b$ toroidal equipotential, which is convex everywhere. The inner surfaces are irrelevant to the electrostatic properties.

On the symmetry axis $\rho = 0$,

$$Z \to (b^2 + az + z^2)(b^2 - az + z^2)[(b^2 + z^2)^2 + a^2 b^2]. \tag{8}$$

Thus the tori meet the symmetry axis at the four points $z = \pm\frac{1}{2}(a \pm \sqrt{a^2 - 4b^2})$. When $a > 2b$, the four points are real. There is an inner surface, looking like a flying saucer. Since all excess charge resides on the outside of a closed conductor, these inner surfaces do not affect the electrostatic properties. At $a = 2b$, the four points coalesce to two, and the torus has a conical indentation on the symmetry axis at $z = \pm b$. For $a < 2b$, the four points are all imaginary: the tori are open and do not meet the axis. These open tori do not form a family of physical equipotential surfaces (see problem 4 of section 6).

Finally, in the $z = 0$ plane, we have

$$Z \to (b^2 - \rho^2)^3 (a^2 + b^2 - \rho^2). \tag{9}$$

Thus the tori outer radii (in the $z = 0$ plane) are given by $\rho_{\max} = \sqrt{a^2 + b^2}$.

In spherical polar coordinates (r, θ), where $r^2 = x^2 + y^2 + z^2 = \rho^2 + z^2$ and θ is the polar angle, with $\rho = r \sin\theta$ and $z = r\cos\theta$,

$$\begin{aligned} Z = 16b^4 r^4 \cos^4\theta + b^2 r^2 [8(r^2 - b^2)^2 - 4a^2(r^2 - b^2) - a^4] \cos^2\theta \\ + (r^2 - b^2)^3 [r^2 - b^2 - a^2]. \end{aligned} \tag{10}$$

This is a quadratic in $\cos^2\theta$, so we can solve for $\theta(a, b; r)$.

A more tractable form of (5) results from expanding about the critical circle: we set

$$\rho = b + d\cos\chi, \qquad z = d\sin\chi \tag{11}$$

in expressions (2), which become

$$r_\pm^2 = d[d + 2b\,\mathrm{e}^{\pm\mathrm{i}\chi}]. \tag{12}$$

Then

$$r_+^2 + r_-^2 = 2d[d + 2b\cos\chi], \qquad r_+^2 r_-^2 = d^2[d^2 + 4bd\cos\chi + 4b^2]. \tag{13}$$

With these substitutions, we obtain a quadratic in $\cos\chi$:

$$\begin{aligned} b^2(a^2 - 4d^2)^2\cos^2\chi - 2bd[d^2(3a^2 - 4d^2) + 4b^2(a^2 - 4d^2)]\cos\chi \\ + d^6 + (8b^2 - a^2)d^4 + 4b^2(4b^2 - a^2)d^2 - a^4 b^2 = 0. \end{aligned} \tag{14}$$

The discriminant of the quadratic is

$$4a^6 b^2 (d^4 - 4d^2 b^2 + a^2 b^2), \tag{15}$$

which is non-negative for $a \geqslant 2b$, giving real roots for $\cos\chi$.

A yet simpler representation of the equipotentials is provided by elliptic-hyperbolic (or oblate-spheroidal) coordinates [2, 5, 12]:

$$\rho = b[(1 + \xi^2)(1 - \eta^2)]^{\frac{1}{2}}, \qquad z = b\xi\eta, \tag{16}$$

because these give the complex distances $r_\pm$ defined in (2) free of square roots:

$$r_\pm = b(\xi \mp \mathrm{i}\eta). \tag{17}$$

Thus the potential defined in (3) becomes

$$V(\xi, \eta) = \frac{Q}{b}\frac{\xi}{\xi^2 + \eta^2}. \tag{18}$$

The equipotentials $V = Q/a$ are now given by

$$\xi^2 - \frac{a}{b}\xi + \eta^2 = 0. \tag{19}$$

The range of the coordinates is chosen to be $0 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant 1$, to make V take the sign of Q. Expressing ξ in terms of η on the equipotential $V = Q/a$ gives

$$\xi_\pm = \frac{a}{2b} \pm \sqrt{\left(\frac{a}{2b}\right)^2 - \eta^2} \tag{20}$$

with both roots real for all physical η provided $a \geqslant 2b$. In the (ξ, η) coordinate system, the equipotentials lie on arcs of circles centred on $\left(\frac{a}{2b}, 0\right)$ and with radius $a/2b$, since

$$\left(\xi - \frac{a}{2b}\right)^2 + \eta^2 = \left(\frac{a}{2b}\right)^2. \tag{21}$$

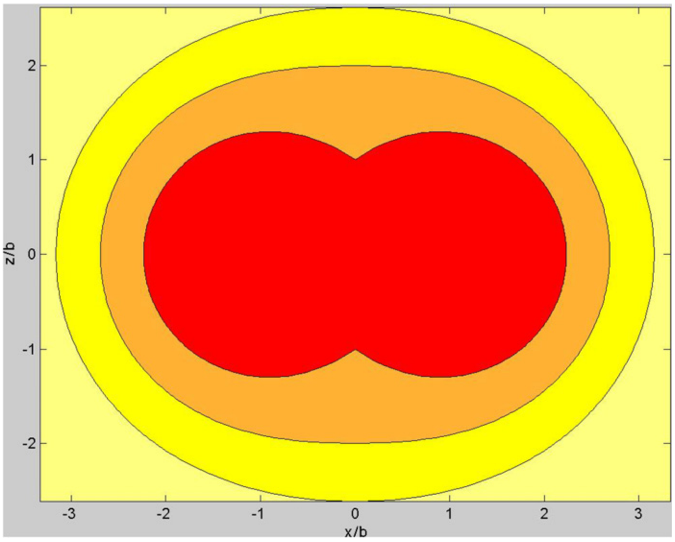

Figure 2. Cross-sections of the $a = 2b$, $a = 5b/2$ and $a = 3b$ equipotential surfaces. (The inner surfaces visible in figure 1 are not shown.) At $a = 2b$, there are conical indentations at the poles, at an angle given by (27). Beyond $a = (5/2)b$, the toroids are convex everywhere.

The equipotentials in (ρ, z) or (r, θ) coordinates follow from the inversion of (16):

$$\begin{aligned} \xi^2 &= \frac{1}{2b^2}\left\{\sqrt{(r^2-b^2)^2+4b^2z^2}+r^2-b^2\right\} \\ \eta^2 &= \frac{1}{2b^2}\left\{\sqrt{(r^2-b^2)^2+4b^2z^2}-(r^2-b^2)\right\}. \end{aligned} \tag{22}$$

Figure 2 shows the equipotential surfaces in cross-section.

3. The capacitance of the tori and their limiting geometrical forms

Let V_0 be the electrostatic potential on a particular torus, characterized by lengths a and b. Since on the torus

$$V(\rho, z) = V_0 = Q/a, \tag{23}$$

the capacitance is independent of b:

$$C = \frac{Q}{V_0} = a. \tag{24}$$

Note that b determines the size of the critical circle, while the outer radius in the $z = 0$ plane is $\sqrt{a^2+b^2}$. If we fix b and let a increase from $2b$, the closed body has radial and axial radii

$$\rho_{\max} = \sqrt{a^2+b^2} \qquad \text{and} \qquad z_{\max} = \tfrac{1}{2}\left(a+\sqrt{a^2-4b^2}\right). \tag{25}$$

Thus, for $a \gg b$, the exterior surface is approximately that of a sphere of radius a. The circumscribed sphere has radius $\sqrt{a^2+b^2}$ (for any a/b) and thus a capacitance greater than a. Likewise, the inscribed sphere has radius $\frac{1}{2}(a+\sqrt{a^2-4b^2})$ when $a > 2b$, and consequently a capacitance less than a.

For $a = 2b$ we have the smallest torus, with radial and axial semi-axes

$$\sqrt{5}b \qquad \text{and} \qquad b. \tag{26}$$

The smallest torus has a conical indentation at the poles, the half-angle of the cone being

$$\arctan(\sqrt{2}) \approx 54.7^\circ. \tag{27}$$

When $a = 5b/2$, the toroid is flat at the poles, which are at distance $2b$ from the origin. For greater values of a, the toroids are convex everywhere.

4. The surface charge density

The electric field is $\boldsymbol{E} = -\nabla V$. At the $V = V_0 = Q/a$ surface, the magnitude of the field (perpendicular to the equipotential) is given by

$$E_\perp^2 = [(\partial_\rho V)^2 + (\partial_z V)^2]_{V=V_0}. \tag{28}$$

In elliptic-hyperbolic coordinates, on using expressions (19) and (20) of [12] to convert the derivatives, we find

$$(\partial_\rho V)^2 + (\partial_z V)^2 = \frac{Q^2}{b^4}\frac{\xi^4 + \xi^2 - 3\xi^2\eta^2 + \eta^2}{(\xi^2 + \eta^2)^4}. \tag{29}$$

Thus on the $V = Q/a$ surface, where (19) holds, we have

$$E_\perp^2 = \frac{Q^2}{a^4 b}\frac{a - 3a\xi^2 + 4b\xi^3}{\xi^3}. \tag{30}$$

The range of ξ is given by setting $\eta^2 = 1$ and $\eta^2 = 0$ in ξ_+, given by (20)

$$\begin{aligned} \xi_{\min} &= \frac{a}{2b} + \sqrt{\left(\frac{a}{2b}\right)^2 - 1} \quad \text{(poles)} \\ \xi_{\max} &= \frac{a}{b} \quad \text{(equator)}\,. \end{aligned} \tag{31}$$

When $a = 2b$, these are 1 and 2 respectively, and $E_\perp^2$ factors:

$$E_\perp^2 \to \frac{Q^2}{b^4}\frac{(\xi - 1)^2(2\xi + 1)}{8\xi^3}. \tag{32}$$

This shows that the electric field (and thus the surface charge density $\sigma = E_\perp/4\pi$) goes to zero at the conical polar indentation when $a = 2b$. There is an analogous situation in the equipotentials of two like charges, with a real displacement between them, discussed by Maxwell [13]: 'There is one equipotential surface . . . which consists of two lobes meeting at the conical point P. That point is a point of equilibrium, and the surface-density of a body of the form of this surface would be zero at this point'.

Let $\mathrm{d}s = \sqrt{(\mathrm{d}\rho)^2 + (\mathrm{d}z)^2}$ be an element of arc length on a central section of an equipotential surface, as shown in figure 2. The area associated with this arc, obtained by rotation around the z-axis by 2π, is $\mathrm{d}A = 2\pi\rho\,\mathrm{d}s$. On an equipotential, we have $\mathrm{d}V = 0 = (\partial_\rho V)\,\mathrm{d}\rho + (\partial_z V)\,\mathrm{d}z$, so $\mathrm{d}z/\mathrm{d}\rho = -\partial_\rho V/\partial_z V$, and

$$\mathrm{d}A = 2\pi\rho\sqrt{(\mathrm{d}\rho)^2 + (\mathrm{d}z)^2} = 2\pi\rho\,\mathrm{d}\rho|\partial_z V|^{-1}\sqrt{(\partial_\rho V)^2 + (\partial_z V)^2}. \tag{33}$$

The charge density is $\sigma = E_\perp/4\pi$ by Gauss's law, so

$$\mathrm{d}Q = \rho\,\mathrm{d}\,\rho|\partial_z V|^{-1}[(\partial_\rho V)^2 + (\partial_z V)^2]. \tag{34}$$

(We have multiplied by 2 to get the total charge, since (33) gives the area for the upper ($z > 0$) part of the equipotential surface only.) In terms of the ξ coordinate, the element of charge is

$$\mathrm{d}Q = Q\frac{a - 3a\xi^2 + 4b\xi^3}{2a\xi\sqrt{\frac{a}{b}\xi - \xi^2}}. \tag{35}$$

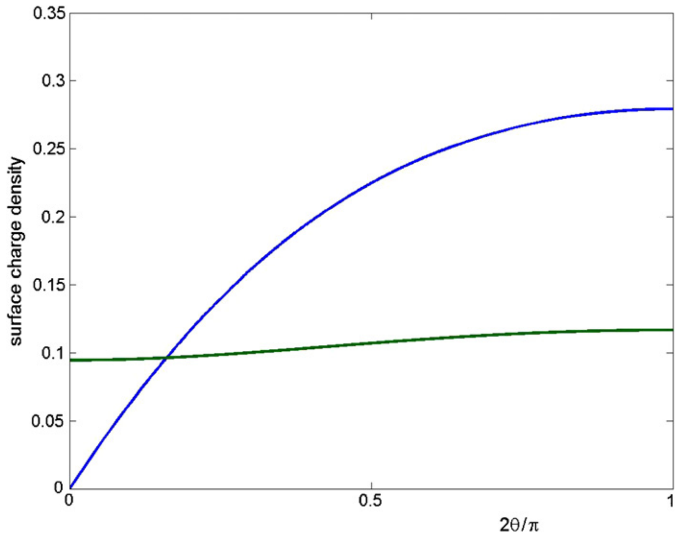

Figure 3. The surface charge density on the $a = 2b$ and $a = 3b$ equipotential surfaces, plotted versus the polar angle θ. Note the zero density at $\theta = 0$ (the cone vertex) when $a = 2b$. The surface charge density is in units of $Q/4\pi b^2$.

The total charge is Q, as it must be. This follows from the identity

$$\int_{\frac{a}{2b}+\sqrt{\left(\frac{a}{2b}\right)^2-1}}^{a/b} \mathrm{d}\xi \, \frac{a - 3a\xi^2 + 4b\xi^3}{2a\xi\sqrt{\frac{a}{b}\xi - \xi^2}} = 1 \qquad (a \geqslant 2b). \tag{36}$$

To prove (36), we change to η as the integration variable and then set $\eta = \alpha \sin\psi$, where $\alpha = a/2b$. The result is

$$\frac{1}{2}\int_0^{\psi_m} \mathrm{d}\psi [\alpha(2\cos\psi - 1)(1+\cos\psi) - \alpha^{-1}(1+\cos\psi)^{-1}]$$
$$= \frac{1}{2\alpha}\tan\left(\frac{\psi_m}{2}\right)\left[1 + 4\alpha^2\cos^4\left(\frac{\psi_m}{2}\right)\right], \tag{37}$$

where $\psi_m = \arcsin(1/\alpha)$, and this reduces to unity on the substitution $\tan(\psi_m/2) = \alpha - \sqrt{\alpha^2 - 1}$.

Figure 3 shows the surface charge density on the $a = 2b$ and $a = 3b$ equipotential surfaces. The polar angle θ is related to the elliptic-hyperbolic coordinates by

$$\sin\theta = \frac{\rho}{r} = \left[\frac{(1+\xi^2)(1-\eta^2)}{1+\xi^2-\eta^2}\right]^{\frac{1}{2}}. \tag{38}$$

On the surface $V = Q/a$, determined by (19), we therefore have

$$\sin\theta = \left[\frac{(1+\xi^2)\left(1 - \frac{a}{b}\xi + \xi^2\right)}{1 - \frac{a}{b}\xi + 2\xi^2}\right]^{\frac{1}{2}}. \tag{39}$$

5. Summary and discussion

The surfaces of constant potential due to two equal charges, displaced from the origin by opposite imaginary translations along the z-axis, form a family of toroids, characterized by two lengths. Of these, a determines the potential value and thus the capacitance. The other length fixes the critical circle radius $\rho = b$. The physical tori, for $a \geqslant 2b$, correspond to equipotentials, which approximate spheres for $a \gg b$. The spatial extent of the tori is $\sqrt{a^2 + b^2}$ radially and $\frac{1}{2}(a + \sqrt{a^2 - 4b^2})$ axially. Any one of the equipotentials can represent a conducting body. For all $a \geqslant 2b$, the capacitance is exactly equal to a, the same as that of a sphere of radius a.

The idea of an imaginary translation is not new. The fact that the Laplacian is invariant to translations is well known. Deschamps [14] transformed the spherically diverging solution $\exp(\mathrm{i}kr)/r$ of the Helmholtz equation to a function representing a beam by means of an imaginary translation. The results and their developments are not without problems: see, for example, [12] for references and a discussion.

6. Suggested class problems

Problem 1. *Electric field and surface charge density near the conical indentation*

It is known that the potential at a conical indentation in a charged conductor is V_0 plus a term proportional to $r^\nu P_\nu(\cos\theta)$, where the order ν is determined by $P_\nu(\cos\beta) = 0$, β being the half-angle of the cone (see, for example, [15, section 3.4]). In this context, r and θ are spherical polar coordinates relative to the apex of the cone as origin. For the $a = 2b$ torus, the half-angle is (verify)

$$\beta = \arctan(\sqrt{2}) = \arccos\frac{1}{\sqrt{3}}. \tag{40}$$

Thus $\nu = 2$, remembering that $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$. Equation (32) gives $E_\perp$ for the $a = 2b$ torus, with $\xi = 1$ corresponding to the apex, and is linear in $\xi - 1$, to lowest order. Show that this is consistent with $E_\perp$ being proportional to r. (Hint: expand r^2 in powers of $\xi - 1$ and show that the leading term is $3b^2(\xi - 1)^2$.)

Problem 2. *Volume and capacitance*

Szegö [16] has proved the Poincaré conjecture that of all bodies of a given volume, the sphere has the smallest capacitance:

$$C \geqslant (3V/4\pi)^{\frac{1}{3}}. \tag{41}$$

The volume of the toroids may be found by integrating

$$\mathrm{d}V = (2z)(2\pi\rho\,\mathrm{d}\rho) \tag{42}$$

(the integration is easier in elliptic-hyperbolic coordinates). The result, expressed in terms of $\alpha = a/2b$, is

$$V = \frac{4}{3}\pi a^3\left\{\frac{1}{2} + \frac{9}{32}\alpha \arcsin\left(\frac{1}{\alpha}\right) + \frac{1}{16}\sqrt{\alpha^2 - 1}\left(\frac{7}{2\alpha} + \frac{1}{\alpha^3}\right)\right\}. \tag{43}$$

Compare the inequality (41) with the exact result $C = a$, as a function of α. Show that the right-hand side of (41) divided by a is

$$1 - \tfrac{1}{80}\alpha^{-4} + O(\alpha^{-6}) \tag{44}$$

at large α and $\left\{\frac{1}{2} + \frac{9}{32}\arcsin(1)\right\}^{\frac{1}{3}} \approx 0.98$ at $\alpha = 1$. Also, the contents of the curly bracket in (43) increase monotonically with α. Thus the Poincaré–Szegö bound differs from the exact capacitance by at most 2% for the tori.

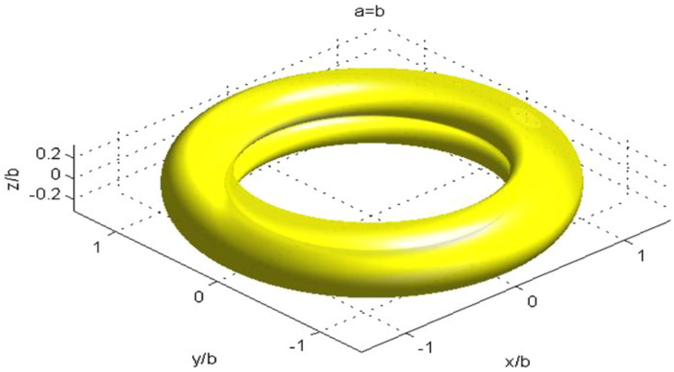

Figure 4. The $a = b$ torus. Note the cusp on the inner surface, centred on the critical circle, and discussed in problem 4 in section 6.

Problem 3. *Surface area and capacitance*

Szegö has also shown that, at least for convex bodies,

$$C \leqslant \frac{A^2}{12\pi V}, \tag{45}$$

where A is the surface area [17]. The element of the surface area is given by (33), in which the partial derivatives of the potential $V(\rho, z)$ are to be evaluated at $V = V_0$. I have not found a closed form for the surface area of the tori in general, but when $a = 2b$ the area $A(\alpha)$ is

$$A(1) = 4\pi a^2 \frac{3}{32}\left\{5 \arctan\sqrt{\frac{2}{3}} + 3\sqrt{6}\right\}, \tag{46}$$

which is approximately 1.01 times $4\pi a^2$. From (43), the corresponding volume at $\alpha = 1$ is

$$V(1) = \frac{4}{3}\pi a^3 \left\{\frac{1}{2} + \frac{9\pi}{64}\right\}, \tag{47}$$

which is about 94% of the volume of a sphere of radius a. With these values, you can evaluate the right-hand side of (45) for the least spherical ($a = 2b$) torus. Does the inequality still hold, even though only tori with $a \geqslant 5b/2$ are convex everywhere?

Problem 4. *The inner cavities and the open tori*

We have concentrated on the surfaces which are physical equipotentials, namely the closed tori with $a \geqslant 2b$. When a is less than $2b$ the equations still have solutions (which are now open tori) but which cannot represent nested equipotentials, because these tori all meet at a cusp on the critical circle $\{\rho = b, z = 0\}$. To explore the cusp, it is convenient to use the polar coordinates relative to the critical circle, introduced in (11). On the critical circle $d = 0$, so from (14) we see that $\cos^2\chi = 1$. Show that near the critical circle $d \to -a^2 \sin^2\chi/8b\cos\chi$ and hence $\chi \to \pi$. Setting $\chi = \pi - \psi$ shows that

$$b - \rho \approx \frac{a^2}{8b}\psi^2, \qquad |z| \approx \frac{a^2}{8b}\psi^3, \tag{48}$$

so that there is an inward cusp at $\{\rho = b, z = 0\}$, where

$$|z| \approx \frac{\sqrt{8b}}{a}(b-\rho)^{\frac{3}{2}}, \qquad b - \rho \approx \left(\frac{a^2}{8b}\right)^{\frac{1}{3}} |z|^{\frac{2}{3}}. \tag{49}$$

(This cusp is also there for the inner cavities of the $a \geqslant 2b$ closed tori.) As a decreases, the tori become slimmer; when a is much smaller than b, show (by expanding in powers of a/b) that the tori cross-sections take the cardiod form

$$d = \frac{a^2}{4b}[1 + \cos\chi]. \tag{50}$$

Figure 4 shows the $a = b$ torus.

Acknowledgment

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