Analysis of Lloyd’s mirror fringes for graded-index reflectors

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Measurement of the fringes produced by interference between direct and reflected light in the Lloyd mirror configuration can give information about the refractive-index profile of the reflecting surface, as has recently been shown. The experiment is performed near grazing incidence, and the analysis given so far has been based on the short-wave approximation, which is known to fail at grazing incidence, and no account has been taken of the difference between s and p polarizations. I give exact reflection phases for the solvable exponential refractive-index profile for both polarizations and develop general results for an arbitrary smooth profile that are valid near grazing incidence. I find that, for profiles that are slowly varying on the scale of the wavelength, the path-integral approximation to the phase is accurate except for the first few interference fringes, which come from glancing incidence reflections. The s and p phases are very nearly equal under the same conditions. Explicit formulas are given for dielectric function profiles that are linear or quadratic in the depth. The characteristic length of the quadratic profile may be found from the variation in the fringe spacing.

1. INTRODUCTION

Klein, Opat, and collaborators have suggested 1 and implemented 2,3 a method of obtaining information about surfaces by measurement of the interference fringes produced in the Lloyd mirage configuration, so called because total reflection by a stratified medium near grazing incidence is precisely the mirage-producing situation. This method is one possible implementation of the holographic reconstruction idea put forward by Smith. 4 The analysis given 2,3 and the inversion method of Allen and Lipperheide 5 use the semiclassical short-wave approximation, in which the phase difference between the direct and the reflected waves is calculated along ray paths, as shown in Fig. 1.

The direct ray in Fig. 1 has length \( L_1^2 + (H - h)^2 \) and lies entirely in medium 1 of index \( n_1 \). The reflected ray has two straight parts that lie in medium 1, of lengths \( \sqrt{L_1^2 + H^2} \) and \( \sqrt{L_2^2 + h^2} \), plus a curved part along which the phase increment is, in the ray approximation,

\[
\Delta \xi \approx \frac{\omega}{c} \int_0^{z_0} \frac{n^2(z) \, dz}{\sqrt{n^2(z) - n^2(z_0)}} - \pi/2. \tag{1}
\]

A derivation of relation (1) is given below (see also Refs. 6 and 7); \( z_0 \) is the turning point at which the wave vector component perpendicular to the stratification (the \( z \) component) becomes zero. The transverse \( (x) \) component of the wave vector is constant by transversal invariance in the \( x \) direction, which implies Snell’s law,

\[
n \cos \theta_1 = n(z) \cos \theta(z) = n(z_0), \tag{2}
\]

where \( \theta_1 \) is the glancing angle of incidence and \( \theta(z) \) is the glancing angle within the variable-index medium. Equation (2) defines \( z_0 \), which we see is a function of the angle of incidence. (We assume here that the refractive-index profile decreases monotonically with \( z \), so that there is at most one turning point at each angle of incidence.) The normal component \( q(z) \) of the wave vector is zero at the turning point:

\[
q(z) = -\frac{\omega}{c} n(z) \sin \theta(z) = \pm \frac{\omega}{c} \sqrt{n^2(z) - n^2(z_0)} \tag{3}
\]

[the second equality follows from Eq. (2)]. The phase increment along the curved part of the ray is \( \int k \cdot ds \) where \( k \) is the wave vector and \( ds \) is an element of path length. This can be decomposed into contributions from motion in the \( x \) and \( z \) directions: let

\[
K = \frac{\omega}{c} n_1 \cos \theta_1 = -\frac{\omega}{c} n(z) \cos \theta(z) = \frac{\omega}{c} n(z_0) \tag{4}
\]

be the constant \( x \) component of the wave vector and (again in the ray approximation)

\[
\Delta x \approx 2 \int_0^{z_0} \frac{dz}{\tan \theta(z)} = 2n(z_0) \int_0^{z_0} \frac{dz}{\sqrt{n^2(z) - n^2(z_0)}} \tag{5}
\]

be the transverse displacement of the ray in the graded-index medium. Then

\[
\Delta \xi = K \Delta x + 2 \int_0^{z_0} q(z) \, dz - \pi/2, \tag{6}
\]

which reduces to relation (1). The extra term \(-\pi/2\) is a correction to the ray approximation, valid at short wavelengths for all dielectric function profiles that are smooth enough to be approximated by a linear variation in the neighborhood of the turning point [see, for example, Ref. 7, Sec. 6–7 and Eqs. (10.11)–(10.14)].

The phase difference between the reflected and the direct rays is
2. Comparison of exact results with the short-wave phase formula

The interference pattern on the screen in Fig. 1 can in principle be found by solving the diffraction problem with a line source (a slit) and a gradient-index boundary. The ray approximation is the classical limit, and the short-wave result [relation (1)] for \( \Delta \), approximates the ray locally by a plane wave with wave vector along the ray direction. In the formulas (1) and (6) no distinction is made between the s and p polarizations (i.e., between the TE and TM waves), and only is made of short-wave results near grazing incidence, where it is known that they fail (Ref. 7, p. 133). These approximations will be examined in this paper.

We will still assume that the cylindrical waves diverging from the line source can be approximated by plane waves. The term \( \exp(iKx) \) represents the motion in the x direction throughout the stratification, giving a phase shift \( K\Delta x \) for the curved part of the ray; we will therefore concentrate on the wave motion in the z direction. If the incoming plane wave is \( \exp(iq_1 z) \), the reflected wave will be (for s polarization) \( r_s \exp(-iq_2 z) \) in medium 1. In the total reflection situation (\( n_1 > n_2, \cos\theta_1 > n_2/n_1 \)), \( r_s = \exp(i\delta_s) \), where in the short-wave approximation

\[
\delta_s \approx \delta_a = 2 \int_0^{z_0} q(z)dz - \pi/2
\]

\[
= 2 \frac{\omega}{c} \int_0^{z_0} dz \sqrt{n^2(z) - n^2(z_0)} - \pi/2.
\]

We assume that the refractive index decreases monotonically from \( n_1 \) at \( z = 0 \) to \( n_2 \) at \( z = \Delta x \). Then at grazing incidence (\( \theta_1 \to 0 \)) the turning point \( z_0 \) tends to zero, and the short-wave approximation to \( r_s \) tends to \(-i\). This is wrong; it is known that at grazing incidence \( r_s \) always tends to \(-1 \) for all stratifications (Ref. 7, Sec. 2–3). Let \( F(z) \) and \( G(z) \) be two linearly independent solutions of the s-wave equation

\[
\frac{d^2E}{dz^2} + q^2(z)E = 0
\]

in the range of variable index, namely, \( 0 \leq z \leq \Delta x \). Then, from Eq. (2.25) of Ref. 7,

\[
r_s = q_1 q_2 (F, G) + i q_1 (F, G') + i q_2 (F', G) - (F', G') \quad q_1 q_2 (F, G) + i q_1 (F, G') - i q_2 (F', G) + (F', G'),
\]

where

\[
(F, G) = F_1 G_2 - G_1 F_2, \quad (F', G') = F'_1 G_2' - G'_1 F_2',
\]

\[
(F', G) = F'_1 G_2' - G'_1 F_2', \quad (F', G') = F'_1 G_2',
\]

and \( F_1 = F(0^+), F_2 = F(\Delta x^-) \). \( F_1' \) is the derivative of \( F(z) \) evaluated at \( z = 0^+ \), etc.

It is immediately clear from Eq. (10) that \( r_s \to -1 \) as \( q_1 \to 0 \). When \( n_1 > n_2 \) and \( \cos\theta_1 > n_2/n_1 \), the normal component of the wave vector in medium 2 is imaginary:

\[
q_2 = c \sqrt{n_2^2 - n_1^2 \cos^2\theta_1} = i \frac{c}{\sqrt{n_2^2(z_0) - n_2^2}} = i |q_2|.
\]

In the total reflection situation the reflection amplitude thus takes the form

\[
r_s = \frac{q_1 [(F, G') + |q_2|(F, G)] + i[(F', G') + |q_2|(F', G)]}{q_1 (F, G') + |q_2|(F, G)} - i[(F', G') + |q_2|(F', G)].
\]
The wave equation (9) is linear, with real coefficients, and thus the functions $F(z)$ and $G(z)$ may be taken to be real. Then Eq. (13) implies that

$$
\delta_s = 2 \arctan \left( \frac{(F', G') + |q_2(F', G')|}{q_1(F', G') + |q_2(F, G)|} \right). \tag{14}
$$

We see again that $r_s \to -1$ as $q_1 \to 0$, since $\delta_s \to \pm \pi$ at grazing incidence.

A comparison of the approximate $\delta_s$ [as given by relation (8)] with the exact phase [Eq. (14)] of the reflection amplitude has been given for the hyperbolic tangent profile in Fig. 6–9 of Ref. 7. Here we consider a refractive-index profile that is solvable for both $s$ and $p$ polarizations, namely, the exponential profile (see Sec. 2–5 of Ref. 7)

$$
n(z) = \begin{cases} 
    n_1 & z < 0 \\
    n_1 \exp(z/b) & 0 \leq z \leq \Delta z, \\
    n_2 & \Delta z > z
\end{cases} \tag{15}
$$

where $b = \Delta z/\log(n_2/n_1)$. The $s$-polarization solutions are Bessel functions $F(z) = J_s(u)$ and $G(z) = Y_s(u)$, with argument $u$ and order $s$ given by

$$
u = \frac{n_1 \omega}{c} |b| \exp(z/b), \quad s = K |b| = n(z) \frac{n_1 \omega}{c} |b| \tag{16}
$$

Note that the argument and the order are equal at the classical turning point $z = z_0$.

For $p$ polarization the solutions are $u J_p(u)$ and $u Y_p(u)$, where

$$
p^2 = (Kb)^2 + 1 = s^2 + 1. \tag{17}
$$

From Eq. (2.98) of Ref. 7, we find that the $p$ reflection phase is

$$
\delta_p = 2 \arctan \left( \frac{(F', G')/u_1 + (F', G)/u_2 + (F, G)/u_1 u_2 + |q_2|(F', G) + (F, G)/u_1}{q_1(F', G') + (F, G)/u_2 + |q_2|(F, G)} \right). \tag{18}
$$

where now $F(z) = J_p(u)$ and $G(z) = Y_p(u)$. [From Eqs. (2.26) and (2.27) of Ref. 7 we see that the reflection amplitude for the normal component ($E_x$) of the electric field is $-r_p$ and that for the tangential component ($E_y$) it is $r_p$. At grazing incidence the normal component is dominant for the $p$ wave, so the phases of $r_s$ and $-r_p$ apply to the $E_y$ and $E_z$ components. Equation (18) gives the phase of $-r_p$.]

For the exponential profile, $\delta_s$ may be evaluated analytically. We find that

$$
\delta_s = 2 n_1 \frac{\omega}{c} |b| \sin \theta_1 - \theta_1 \cos \theta_1 - \pi/2, \tag{19}
$$

where $\theta_1$ is the glancing angle of incidence and $n_1 \cos \theta_1 = n(z_0)$. This formula applies in the total reflection region, where $\theta_1 < \arccos(n_2/\chi_1)$. Note that $\delta_s$ approaches $-\pi/2$ at grazing incidence, with $\delta_s + \pi/2$ tending to zero as $\frac{2}{n_1} (\omega/c) |b| \theta_1^3$. The proportionality of the phase integral to the cube of the glancing angle of incidence near grazing incidence is universal for profiles that have a finite derivative at $z = 0$: From $n_1 \cos \theta_1 = n(z_0)$ and $n(z) \approx n_1 - z |d n/dz|_{z=0}$ we find that

$$
\delta_s = 2 n_1 \frac{\omega}{c} |b| \sin \theta_1 - \theta_1 \cos \theta_1 - \pi/2, \tag{19}
$$

where $\theta_1$ is the glancing angle of incidence and $n_1 \cos \theta_1 = n(z_0)$. This formula applies in the total reflection region, where $\theta_1 < \arccos(n_2/\chi_1)$. Note that $\delta_s$ approaches $-\pi/2$ at grazing incidence, with $\delta_s + \pi/2$ tending to zero as $\frac{2}{n_1} (\omega/c) |b| \theta_1^3$. The proportionality of the phase integral to the cube of the glancing angle of incidence near grazing incidence is universal for profiles that have a finite derivative at $z = 0$: From $n_1 \cos \theta_1 = n(z_0)$ and $n(z) \approx n_1 - z |d n/dz|_{z=0}$ we find that

$$
z_0 = \frac{n_1 \theta_1^2}{2 |dn/dz|_{z=0}} + O(\theta_1^4) \tag{20}
$$

and thus that

$$
\int_{z_0}^{z} dz q(z) = \frac{\omega}{c} \int_{z_0}^{z} dz \sqrt{n^2(z) - n^2(z_0)} = \frac{\omega}{c} n_1^2 \left( \frac{1}{|dn/dz|_{z=0}} \right) \frac{\theta_1^3}{3} + O(\theta_1^5). \tag{21}
$$

For such profiles, Eq. (20) shows that the Lloyd mirror experiment near grazing incidence probes only depths of order $\theta_1^2/(d \log n(z)/dz)_{z=0}$.

More diffuse profiles can have zero first derivative at $z = 0$, and $n(z) \approx n_1 - 1/2 z^2 [d^2 n/dz^2]_{z=0}$, in which case

$$
z_0 = \left( \frac{n_1}{|d^2 n/dz^2|_{z=0}} \right)^{1/2} \theta_1 + O(\theta_1^3), \tag{22}
$$

and the phase integral is independent of the second derivative to leading order in $\theta_1$:

$$
\int_{z_0}^{z} \frac{q(z)}{dz} = \frac{\pi}{4} \frac{\omega}{c} n_1 \theta_1 + O(\theta_1^3). \tag{23}
$$

The short-wave approximation $\delta_s$ to the $s$ wave reflection phase $\delta_s$ is compared with the exact phase in Fig. 2. We see that the short-wave approximation fails at grazing incidence but becomes accurate away from grazing incidence in cases (such as the example used here) in which the profile is smooth and thick on the scale of the wavelength. Note also that $\delta_s$ and $\delta_p$ agree closely in these circumstances (Appendix A gives the relationship between $\delta_s$ and $\delta_p$ for the exponential profile).

The exact $s$-wave phase is given by Eq. (14) and is seen to depend on solutions of the wave equation, and their derivatives, evaluated at the boundaries of the inhomogeneous reflecting region. The approximate $s$-wave phase $\delta_s = 2 \int_{z_0}^{z} q(z) dz - \pi/2$ integrates the phase increment $q(z) dz$ only up to the turning point $z_0$. It takes no account of the evanescent penetration of the wave into the region beyond the classical turning point. It might be thought that the use of the full JWKB wave functions in Eq. (14) would lead to a phase $\delta_{sw}$, which does not have the deficiencies of $\delta_s$. We find, however, that although the resultant limiting value $r_s \to -1$ is in accord with the general theorem of Ref. 7 for reflection at grazing incidence, the improved short-wave phase $\delta_s$ fails at small glancing angles, just as the simpler phase $\delta_s$ did. Neither $\delta_s$ nor $\delta_{sw}$ gives the correct linear variation of phase with glancing angle, which will be shown to be a general property in Section 3. Both fail because of the application of short-wave wave functions to a situation in which the problem has a long-wave character at glancing inci-
Since question wave approximations fail. To solve wave equation (9) we utilize solutions for which $q_{1} \to 0$, even when $(o\omega/c)\Delta z = 2\pi \Delta z/\lambda_{\text{vac}}$ is large.

3. REFLECTION PHASE NEAR GRAZING INCIDENCE

In this section it will be shown that the phase for small glancing angle $\theta_{1}$ is linear in $\theta_{1}$, and the coefficient of $\theta_{1}$ will be evaluated for two classes of refractive-index profiles. Let us write Eq. (14) as

$$\delta_{s} = 2 \arctan(Q/q_{1}), \quad Q = \left(\frac{F'}{G'} + \frac{|q_{2}|(F', G)}{(F, G')}\right).$$

(24)

From \arctan(X) $= \text{sgn}(X)\pi/2 - X^{-1} + O(X^{-3})$ we have

$$\delta_{s} = \text{sgn}(Q_{0})\pi - 2q_{1}/Q_{0} + O(q_{1}/Q_{0})^{3},$$

(25)

where $Q_{0}$ is the grazing incidence limiting value of $Q$. Since $q_{1} = n_{c}(o\omega/c)\sin \theta_{1}$, Eq. (25) shows that $\delta_{s}$ is linear in $\theta_{1}$ small glancing angles, if $Q_{0}$ exists.

Next we consider profiles such as the exponential profile of Eq. (15) that are linear in $z$ for small $z$. Near grazing incidence the normal component of the wave vector, $q(z) = n(z)(o\omega/c)\sin \theta(z)$, is small, and the shortwave approximations fail. To solve wave equation (9) we assume that $n(z)$, and therefore also the dielectric function $\varepsilon(z) = n^{2}(z)$, are linear in $z$ for small $z$. Near the turning point $z_{0}$ we have

$$q^{2}(z) \approx (z - z_{0})\frac{\omega^{2}}{c^{2}} \frac{dz}{dz}_{z_{0}}.$$  

(26)

Define a dimensionless variable $\zeta$ by

$$\zeta = (z - z_{0})k_{0}, \quad k_{0}^{3} = \frac{\omega^{2}}{c^{2}} \left(-\frac{dz}{dz}\right)_{z_{0}}.$$  

(27)

Then wave equation (9) transforms to

$$\frac{d^{2}E}{dz^{2}} - \zeta E = 0,$$

(28)

the solutions of which are the Airy functions $Ai(\zeta)$ and $Bi(\zeta)$. Thus $F(z) \sim Ai(\zeta)$, $G(z) \sim Bi(\zeta)$ are two linearly independent approximate solutions of the wave equation. These solutions are accurate near the turning point $z_{0}$, which is at small depths $z$ near grazing incidence. The general formula (14) for $\delta_{s}$ requires evaluation of $F$ and $G$ and of their derivatives at $z = 0$ and $z = \Delta z$; the corresponding values of $\zeta$ are

$$\zeta_{1} = -z_{0}k_{0}, \quad \zeta_{2} = (\Delta z - z_{0})k_{0}.$$  

(29)

For all profiles, $z_{0} \to 0$ at grazing incidence, so we are interested in the values of $Ai(\zeta)$ and $Bi(\zeta)$ and of their derivatives at $\zeta = 0$ and $\zeta = \Delta z k_{0}$. We assume that the profile characteristics and the wavelength of the light are such that $\Delta z k_{0} \gg 1$; from the Airy function asymptotic forms

$$Ai(\zeta) \sim 1/2\pi \zeta^{-1/4} \exp(-2/3\zeta^{3/2}),$$

$$Bi(\zeta) \sim \pi^{1/2} \zeta^{-1/4} \exp(2/3\zeta^{3/2}),$$

(30)

it follows that $Bi$ dominates at the base of the inhomogeneous layer. The fact that the Airy functions do not provide an accurate solution for large $x$ except for the profiles for which $\varepsilon(x)$ is linear in $z$ at all $z$, for which the Airy functions are exact solutions does not matter: the solution that is accurately linear near grazing incidence is $Bi(\zeta)$ at small $\zeta$ will dominate at large $\zeta$ in all cases and cancel out from $\delta_{s}$, as we shall see. At the turning point where $\zeta = 0$ we use the values

$$c_{1} = Ai(0) = Bi(0)/\sqrt{3} = [3^{2/3} \Gamma(2/3)]^{-1},$$

$$c_{2} = -Ai'(0) = Bi'(0)/\sqrt{3} = [3^{1/3} \Gamma(1/3)]^{-1}.$$  

(31)

From Eqs. (30) and (31), neglecting terms of order $\exp[-4/3(\Delta z k_{0})^{3/2}]$, we have

$$Q_{0} \approx -\frac{c_{2}k_{0}}{c_{1}} = \frac{1/3 \Gamma(2/3)}{\Gamma(1/3)}k_{0} \approx -0.729k_{0}.$$  

(32)

We note that this negative value of $Q_{0}$ applies to all profiles for which $n(z)$ decreases linearly with $z$ for small $z$. All such profiles therefore have linear variation of the reflection phase with the glancing angle:

$$\delta_{s} = -\pi + \left(\frac{2c_{1}}{c_{2}}\right) \frac{\omega n_{1}}{ck_{0}} \theta_{1} + O(\theta_{1}^{3}).$$  

(33)

For exponential profile (15) the dimensionless parameter $\omega n_{1}/ck_{0}$ takes the grazing incidence value

$$\frac{\omega n_{1}}{ck_{0}} \to \frac{n_{1}}{2 \log(n_{1}/n_{2})}.$$  

(34)

Figure 2 compared the $s$ wave phase predicted by relations (33) and (34) with the exact phase for the exponential profile.

Another important class of profiles has $n(z)$ or $\varepsilon(z)$ quadratic in $z$ at small $z$: Let

$$\varepsilon(z) = n_{1}(1 - z^{2}/D^{2})$$  

(35)
and set
\[ z = a \zeta, \quad a^2 = \frac{D}{2n_1 \omega c}, \quad \alpha = (aq_1)^2. \] (36)

Wave equation (9) then reduces to
\[ \frac{d^2E}{d\xi^2} + (\alpha - \zeta^2/4)E = 0, \] (37)
for which the standard solutions are parabolic cylinder functions[9] \( U(-\alpha, \zeta), V(-\alpha, \zeta) \). The function \( V \) dominates for large \( \zeta \):
\[ U(-\alpha, \zeta) \sim \zeta^{-1/2} \exp(-1/4\zeta^2), \]
\[ V(-\alpha, \zeta) \sim \left( \frac{2}{\pi} \right)^{1/2} \zeta^{-1/2} \exp(1/4\zeta^2). \] (38)

Thus when \( \Delta z/a \gg 1 \) the grazing incidence limit of \( Q_0 \) is
\[ Q_0 \approx \frac{1}{a} \frac{U'(0, 0)}{U(0, 0)} = -\frac{2\Gamma(3/4)}{\alpha \Gamma(1/4)} \approx -0.478/a \] (39)
[see Eqs. (19.3.5) of Ref. 10]. From Eq. (25) it follows that
\[ \delta_s = -\pi + \frac{\Gamma(1/4)}{\Gamma(3/4)} (n_1 \omega D/c)^{1/2} \theta_1 + O(\theta_1^3). \] (40)

For comparison, we rewrite the result [Eq. (33)] for linear variation in the dielectric function, \( \epsilon(z) = \epsilon_1 (1 - z/d + \ldots) \):
\[ \delta_s = -\pi + \frac{2\Gamma(1/3)}{3^{1/2} \Gamma(2/3)} (n_1 \omega D/c)^{1/3} \theta_1 + O(\theta_1^3). \] (41)

Provided that a limiting value of \( Q \) as \( q_1 \to 0 \) exists, the phase of the reflection amplitude is linear in the grazing angle \( \theta_1 \). The coefficient of \( \theta_1 \) is determined by the variation of \( n(z) \) in the outer boundary of the reflecting layer. This may be expected on physical grounds, since the classical turning point \( z_0 \) approaches the outer boundary at grazing incidence, and the wave is evanescent beyond \( z_0 \).

4. LLOYD MIRAGE FRINGE SPACING

We have seen that the approximate short-wave phase \( \delta_0 \) given by relation (8) and the more complete \( \delta_{sw} \) both fail at grazing incidence, though in different ways. Nevertheless, for profiles that are smooth and thick on the scale of the wavelength, the semiclassical phase \( \delta_0 \) is accurate in a large portion of the total reflection region, namely, for \( \theta_1 \leq \theta_1 \leq \theta_s \), where \( \theta_1 = (n_1 \omega D/c)^{-1/3} \) or \( \theta_1 = (n_1 \omega D/c)^{-1/2} \) for linear or quadratic variation of the index at small \( z \), respectively. The interference fringe pattern is determined by the total phase given in Eq. (7), with \( \Delta \approx K \Delta x + \delta_s \) for s-polarized light:
\[ \Delta = \delta_s + K \Delta x + \frac{\omega}{c} n_1 \sqrt{L_1^2 + H^2} + \sqrt{L_2^2 + h^2} \]
\[ = \sqrt{L^2 + (H - h)^2}. \] (42)

From Fig. 1 we see that
\[ H/L_1 = \tan \theta_1 = h/L_2; \quad L_1 + \Delta x + L_2 = L, \] (43)
so that \( h = (L - \Delta x) \tan \theta_1 - H \) and
\[ L_1 = \frac{L - \Delta x}{1 + h/H}, \quad L_2 = \frac{h}{H} \frac{L - \Delta x}{1 + h/H}. \] (44)

The expression in the square brackets in Eq. (42), which is the length of the straight-line parts of the reflected ray minus the length of the direct ray, is equal to
\[ -\Delta x + \frac{2Hh}{L} + \frac{(H^2 + h^2)\Delta x}{2L^2} + O(L^{-3}). \] (45)

Thus, since \( K = n_1 (\omega/c) \cos \theta_1 \),
\[ \Delta = \delta_s + n_1 (\omega/c) \left[ \cos \theta_1 - 1 \right] \Delta x + \frac{2Hh}{L} \]
\[ + \frac{(H^2 + h^2)\Delta x}{2L^2} + O(L^{-3}). \] (46)

The experimental results are photographs of fringes, seen as a function of \( h \), which itself depends nonlinearly on the angle of incidence. Note that the lateral displacement \( \Delta x \) is a function of the angle of incidence: For linear variation of the dielectric function, \( n^2(z) = \epsilon(z) = \epsilon_1 (1 - z/d) \), we find that \( z_0 = d \sin^2 \theta_1 \), and the ray approximation gives
\[ \Delta x \approx 2n(z_0) \int_{z_0}^{z_2} \frac{dz}{\sqrt{\epsilon(z) - \epsilon(z_0)}} = 4d \sin \theta_1 \cos \theta_1, \]
\[ \delta_s \approx \delta_a = -\pi + \frac{4}{3} n_1 (n_1 \omega D/c) \sin^2 \theta_1. \] (47)

For quadratic variation, \( \epsilon(z) = \epsilon_1 (1 - z^2/D^2) \), we have \( z_0 = D \sin \theta_1 \) and
\[ \Delta x \approx \pi D \cos \theta_1, \]
\[ \delta_s \approx \delta_a = -\pi + \frac{\pi}{2} (n_1 \omega D/c) \sin^2 \theta_1. \] (48)

When we put these expressions into Eq. (46) and write \( \theta_1 \) in terms of \( h \) using \( (L - \Delta x) \tan \theta_1 = H + h \),
\[ \theta_1 \text{(linear)} = \frac{H + h}{L} + O(L^{-3}), \]
\[ \theta_1 \text{(quadratic)} = \frac{H + h}{L} \left[ 1 - \frac{\pi D}{L} \right] + O(L^{-3}), \] (49)
we obtain the total phase shift as a function of the height above the surface, \( h \). Each increase of \( \Delta \) by \( 2\pi \) gives an additional fringe; the intensity is proportional to \( 1 + \exp(i\Delta)^2 = 4 \cos^2(\Delta/2) \). We find that
\[ \Delta \text{(linear)} \approx \frac{\pi}{2} + 2n_1 \frac{\omega H h}{c L} + O(L^{-3}), \] (50)
\[ \Delta \text{(quadratic)} \approx \frac{\pi}{2} + 2n_1 \frac{\omega H h}{c L} \
+ \frac{\pi}{2} n_1 \frac{\omega D}{L} \frac{H^2 + h^2}{L^2} + O(L^{-3}). \] (51)
We see that, because of the extra factor of \( \sin \theta_1 \) in both \( \delta_s \) and \( \Delta x \) for the linear profile, the fringe spacing

\[
\Delta h = \frac{2 \pi}{\Delta x / dh}
\]  

(52)

is constant at \( \lambda_1 L/2H \) in the case of the linear profile \( (\lambda_1 = \lambda/n_1, \text{ where } \lambda = 2 \pi c/\omega \text{ is the vacuum wavelength}) \), whereas it is

\[
\Delta h = \frac{\lambda_1 L}{2H + \pi D h / L} = \frac{\lambda_1 L}{2H} \left( 1 - \frac{\pi D h}{2HL} \right)
\]  

(53)

for the quadratic profile. These results indicate that the two prototypical index profiles, respectively linear and quadratic in the depth, can be distinguished in the fringe pattern and that, further, the characteristic length \( D \) of the quadratic variation might be determined from the decrease in the fringe spacing with \( h \).

5. SUMMARY AND CONCLUSION

We have seen that the path-integral formula for the reflection phase fails at grazing incidence but is otherwise accurate for profiles that are slowly varying on the scale of the wavelength. An improved short-wave approximation, which allows for penetration of the evanescent wave into the classically forbidden region, gives the correct phase at grazing incidence but does not give the correct angle dependence.

An exact solution (for both polarizations) of the wave equation for the exponential profile shows that the \( s \) and \( p \) phases are nearly the same under the same conditions that make the path-integral phase accurate.

All profiles give a reflection phase near grazing incidence that varies linearly with the grazing angle. Explicit formulas for this variation are given for linear and for quadratic variation of the dielectric function with depth.

The fringe spacing is found to be constant for linear profiles, equal to Young’s two-slit fringe spacing \( \lambda_1 L/2H \), when the total phase difference between the direct and reflected rays is calculated to order \( L^{-2} \). For the quadratic profile there is a correction to the total phase of order \( L^{-2} \), which could lead to the extraction of the profile characteristic length \( D \), provided that the parameters of the experiment are chosen so as to make \( Dh/HL \) not too small compared to unity.

APPENDIX A: POLARIZATION DEPENDENCE OF THE REFLECTION PHASE

We will consider the \( s \) and \( p \) phases near grazing incidence. From Eqs. (1.26) and (1.27) of Ref. 7 we see that for \( p \) polarization the reflection amplitude is \( -r_p \), for the normal component \( E_z \) of the electric field and that it is \( +r_p \) for \( E_x \). At grazing incidence the component \( E_z \) dominates; in this paper the reflection phases \( \delta_s \) and \( \delta_p \) are accordingly defined by

\[
\Delta h = \frac{2 \pi}{\Delta x / dh}
\]

\[
\Delta h = \frac{\lambda_1 L}{2H + \pi D h / L} = \frac{\lambda_1 L}{2H} \left( 1 - \frac{\pi D h}{2HL} \right)
\]

\[
r_s = |r_s| \exp(i \delta_s), \quad -r_p = |r_p| \exp(i \delta_p),
\]

(A1)

for comparison of the phase of \( E_y \) (for the \( s \) wave) with the phase of the dominant \( p \)-wave component \( E_z \).

The simplest case to consider is that of a sharp transition, \( \delta_s \) for both polarizations will be negligible only if \( n_2 \approx n_1 \).

Next we look at the opposite case of a slow transition between the media of indices \( n_1 \) and \( n_2 \), taking the exponential profile defined in Eq. (15) as a specific example. For \( s \) polarization the solutions are \( F(z) = J_s(u) \) and \( G(z) = Y_s(u) \); these functions and their derivatives are to be evaluated at the end points

\[
u_1 = n_1 |b| \omega/c, \quad u_2 = n_2 |b| \omega/c,
\]

(A7)

where \( b = \Delta x / \log(n_2/n_1) \). The order \( s \) of the Bessel functions is \( s = K[b] = n_1 |b| (\omega/c) \cos \theta_1 \), and at grazing incidence this tends to \( u_2 \). If \( \Delta x \omega/c \gg 1 \) or if \( n_2 \approx n_1 \) (i.e., if the index variation is slow on the scale of the wavelength or if it is small), \( u_1 \) and \( u_2 \) will be large. Also \( n_1 < n_2 \), so \( u_1 > u_2 \); thus at grazing incidence, \( F_2 = J_s(u_2) \) will be small and \( G_2 = Y_s(u_2) \) will be large. From Eqs. (9.3.7) and (9.3.8) of Ref. 11 we find that

\[
F_2 / G_2 \sim -\frac{1}{2} \exp[-2u_1(\alpha - \tan \alpha)] = \sigma,
\]

(A8)

where \( \text{sech} \alpha = n_2/n_1 \). Thus \( G_2 \) dominates exponentially over \( F_2 \), and the same is true for the derivatives \( G_2' \) and \( F_2' \). The general formula (14) then simplifies to

\[
\delta_s = \arctan(F_1'/q_2 F_1) + O(\sigma).
\]

(A9)

Similarly, the \( p \) polarization phase shift, given by Eq. (18), simplifies to
\[ \delta = \arctan(F_1' q_1 F_1) + O(\sigma) + O(u_1^{-1}) + O(u_2^{-1}) \]  
\hspace{1cm} (A10)

Thus the s and p reflection phases differ (near grazing incidence) only by the exponentially small factor \( \sigma \) and by terms of order \( u_1^{-1} \) and \( u_2^{-1} \). Near equality of \( \delta \) and \( \delta_p \) at glancing incidence can similarly be expected for any smooth and slowly varying profile.

Note that the result (A9) together with Bessel asymptotics enables us to check the general formula for the s-polarization phase shift at glancing incidence: From Eqs. (9.3.23) and (9.3.27) of Ref. 11 we find (see also Watson,\(^\text{12}\) p. 232) that

\[ \frac{J'_v(u)}{J_v(u)} = \left( \frac{6^{1/3} \Gamma(2/3)}{v^{1/3} \Gamma(1/3)} \right) + O(v^{-1}). \]  
\hspace{1cm} (A11)

This result used in Eq. (A9), together with Eqs. (31), gives us

\[ \delta = -\pi + \frac{2 \Gamma(1/3)}{6^{1/3} \Gamma(2/3)} (n_1 |b| \omega/c)^{1/3} \theta_1 + O(\sigma) \]
\[ + O(u_1^{-1}) + O(\theta_1^{-3}), \]  
\hspace{1cm} (A12)

in agreement with Eq. (41), since \( d = |b|/2 \). The angular extent of the region where \( \delta \) is linear in \( \theta_1 \) is thus from zero (glancing incidence) to approximately

\[ \theta_1 = (n_1 |b| \omega/c)^{-1/3} = \left[ \frac{\log(n_1/n_2)}{n_1 \Delta x \omega/c} \right]^{1/3}. \]  
\hspace{1cm} (A13)

**REFERENCES**

8. See, for example, E. Merzbacher, Quantum Mechanics, 2nd ed. (Wiley, New York, 1970), Secs. 7.2 and 7.5.
10. J. C. P. Miller, Chap. 19 of Ref. 9.
11. W. J. Olver, Chap. 9 of Ref. 9.