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Non-existence of separable spheroidal beams

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Abstract
We show that $\psi = R(\xi)S(\eta)\,\mathrm{e}^{\mathrm{i}m\phi}$, a product of radial and angular oblate spheroidal functions and an azimuthal factor, cannot represent physical free-space scalar beams. The reason lies in the discontinuity in the longitudinal derivative of ψ in the focal plane, where ψ is not a solution of the Helmholtz equation on the disc $\xi = 0$.

Keywords: spheroidal, beams

(Some figures in this article are in colour only in the electronic version)

It is known that the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0 \tag{1}$$

is separable in the general ellipsoidal coordinate system plus its ten degenerate forms [1]. Of particular interest is the oblate spheroidal coordinate system [1–3], related to cylindrical polar coordinates (ρ, z, ϕ) by

$$\rho = b[(\xi^2+1)(1-\eta^2)]^{\frac{1}{2}}, \qquad z = b\xi\eta, \qquad \phi = \phi, \tag{2}$$

in which the Helmholtz equation (1) becomes (with $\beta = kb$)

$$\left\{\frac{1}{\xi^2+\eta^2}[\partial_\xi(\xi^2+1)\partial_\xi + \partial_\eta(1-\eta^2)\partial_\eta] + \frac{1}{(\xi^2+1)(1-\eta^2)}\partial_\phi^2 + \beta^2\right\}\psi = 0. \tag{3}$$

Separation of the partial differential equation (3) leads to wavefunctions ψ which are products of a 'radial' function $R(\xi)$, an 'angular' function $S(\eta)$, and the azimuthal factor $\mathrm{e}^{\mathrm{i}m\phi}$, with m taking integer values:

$$\psi(\xi,\eta,\phi) = R(\xi)S(\eta)\,\mathrm{e}^{\mathrm{i}m\phi}. \tag{4}$$

Wavefunctions of this form were suggested as exact non-paraxial scalar beams [4]. Kiselev [5] has noted that 'solutions obtained by separation of variables in oblate and prolate spheroidal coordinates . . . have singularities related to particular features of coordinate systems and do not satisfy (the Helmholtz equation) in the whole space'. (See also the comment on [4] in the last paragraph of [6].)

We have previously shown that only a subset of the wavefunctions (4) can represent physical beams [7]. Here we show that none of them can. That does not preclude their use in the large β paraxial case, but then the Gaussian beam

$$\psi_{\mathrm{G}}(\rho,z) = \frac{b}{b+\mathrm{i}z}\exp\left[\mathrm{i}kz - \frac{k\rho^2}{2(b+\mathrm{i}z)}\right] \tag{5}$$

and its Gauss–Laguerre generalizations are simpler to use.

We note in passing that exact non-separable solutions of the Helmholtz equation are known [8–10]. The problem with these is the singularity on the critical circle $\rho = b$, $z = 0$. On removing the singularity by combining waves travelling in opposite directions [11–13], one has the problem of non-physical free-space propagation in both the $+z$ and $-z$ directions. Thus these non-separable solutions are unphysical free-space scalar beams.

Separation of the partial differential equation (3), by substitution of the product wavefunction (4), leads to the radial and angular equations

$$(\xi^2+1)R''(\xi) + 2\xi R'(\xi) + \left[\beta^2\xi^2 + \frac{m^2}{\xi^2+1} - \alpha\right]R(\xi) = 0 \tag{6}$$

$$(1-\eta^2)S''(\eta) - 2\eta S'(\eta) + \left[\beta^2\eta^2 - \frac{m^2}{1-\eta^2} + \alpha\right]S(\eta) = 0. \tag{7}$$

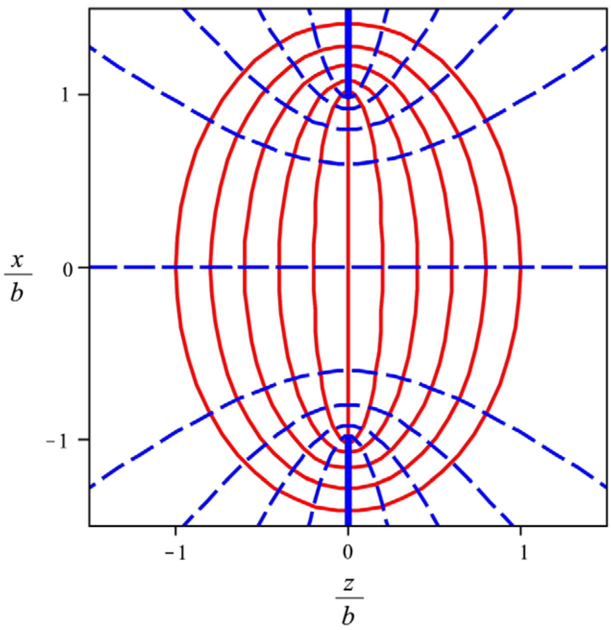

Figure 1. The oblate spheroidal coordinate system. A section in the zx plane is shown. The three-dimensional picture is obtained by rotating the figure about the z-axis. The confocal ellipsoids (solid curves) are surfaces of constant ξ. The central ellipsoid $\xi = 0$ is the disc $\rho \leqslant b$, $z = 0$. The confocal hyperboloids (dashed curves) are surfaces of constant η. In the focal plane $z = 0$ the region outside the disc $\rho \leqslant b$ corresponds to $\eta = 0$ (thick line).

For given $\beta = kb$ and integer m, a set of values $\alpha_{mn}(\beta)$ ($n = 0, 1, 2, \ldots$) of the separation parameter α gives angular solutions $S_{mn}(\beta, \eta)$ finite at $\eta^2 = 1$. These functions have expansions in powers of $1 - \eta^2$, as follows [2, 3]:

$$n - m \text{ even}: \quad S_{mn}(\beta, \eta) = (1-\eta^2)^{m/2} \sum_{\ell=0}^{\infty} C_{2\ell}^{mn} (1-\eta^2)^{\ell} \tag{8}$$

$$n - m \text{ odd}: \quad S_{mn}(\beta, \eta) = \eta(1-\eta^2)^{m/2} \sum_{\ell=0}^{\infty} C_{2\ell}^{mn} (1-\eta^2)^{\ell} \tag{9}$$

(with similar but different expansion coefficients $C_{2\ell}^{mn}$ in the even and odd cases). Thus the angular functions are even or odd in η according to whether $n - m$ is even or odd.

We now consider the properties of oblate spheroidal beam wavefunctions of the form (4), in the focal plane $z = 0$. The oblate spheroidal variables have the ranges $-\infty < \xi < \infty$, $0 \leqslant \eta \leqslant 1$, and the $z = 0$ plane is represented by the disc $\rho \leqslant b$ or $\xi = 0$ and the remainder by $\rho \geqslant b$ or $\eta = 0$ (see figure 1).

We make the following demands on a physical scalar beam wavefunction ψ:

(i) ψ must satisfy the Helmholtz equation (1). (Note that this implies that the derivatives $\partial_z\psi$ and $\partial_\rho\psi$ exist and are continuous: a discontinuity in the first derivative would imply an infinite second derivative, with nothing in the free-space Helmholtz equation to cancel it.)

(ii) The integral $\int_0^\infty \mathrm{d}\rho\rho|\psi|^2$ must be finite ($2\pi\,\Delta z$ times this integral represents the probability of finding a particle in a transverse section of thickness Δz of a Schrödinger particle beam, for example).

(iii) The z-component of the probability density flux J_z, proportional to $\mathrm{Im}(\psi^*\partial_z\psi)$, must be non-zero and finite, at least in the central part of the focal plane (otherwise the wavefunction ψ would not represent a propagating beam). (We could also require that the total flux through any section of the beam be finite, which implies that the integral $\int_0^\infty \mathrm{d}\rho\rho\,\mathrm{Im}(\psi^*\partial_z\psi)$ is finite, but shall not need to.)

The oblate spheroidal wavefunctions

$$\psi_{mn}(\xi, \eta, \phi) = \mathrm{i}R_{mn}^{(3)}(\beta, \xi) S_{mn}(\beta, \eta)\,\mathrm{e}^{\mathrm{i}m\phi} \tag{10}$$

where $n - m$ is odd, and $R^{(3)} = R^{(1)} + \mathrm{i}R^{(2)}$, were shown in [7] to satisfy condition (ii) and also to have physically reasonable isophase surfaces. However, the continuity of the derivative $\partial_z\psi$ is suspect, as is evident in figure 3 of [7]. From equation (19) of [8]

$$\partial_z\psi = \frac{1}{b(\xi^2+\eta^2)}\{\eta(1+\xi^2)\partial_\xi + \xi(1-\eta^2)\partial_\eta\}\psi. \tag{11}$$

Consider $\psi = R(\xi)S(\eta)$ (the factor $\mathrm{i}\mathrm{e}^{\mathrm{i}m\phi}$ in (10) is omitted, to simplify the expressions):

$$\partial_z\psi = \frac{1}{b(\xi^2+\eta^2)}\{\eta(1+\xi^2)R'S + \xi(1-\eta^2)RS'\}. \tag{12}$$

In the focal plane $z = 0$ we have

$$\partial_z(RS) = \begin{cases} \dfrac{S(\eta)}{b\eta}R'(0) & (\xi = 0, \text{ ie } \rho \leqslant b) \\ \dfrac{R(\xi)}{b\xi}S'(0) & (\eta = 0, \text{ ie } \rho \geqslant b). \end{cases} \tag{13}$$

We have already established in [7] that $n-m$ must be odd, so from (9) $S(\eta)/\eta$ will be finite as $\eta \to 0$, and $S'(0)$ will exist. It remains for us to examine the existence of $R'(0)$ and of the limit of $R(\xi)/\xi$ as $\xi \to 0$. Clearly both will exist if $R(\xi)$ is a continuous odd function of ξ. But $R(\xi)$ must be proportional to $R_{mn}^{(1)}(\beta, \xi) + \mathrm{i}R_{mn}^{(2)}(\beta, \xi)$ for a beam propagating in the $+z$ direction, as follows from the asymptotic forms ($\beta\xi \to \infty$) [3]

$$\begin{aligned} R_{mn}^{(1)}(\beta, \xi) &\to \frac{1}{\beta\xi}\cos\left[\beta\xi - \frac{\pi}{2}(n+1)\right] \\ R_{mn}^{(2)}(\beta, \xi) &\to \frac{1}{\beta\xi}\sin\left[\beta\xi - \frac{\pi}{2}(n+1)\right]. \end{aligned} \tag{14}$$

For given m and n, $R^{(1)}$ and $R^{(2)}$ have opposite parities [3], so they cannot both be odd, and thus neither can $R(\xi)$ be odd.

In the example shown in figure 3 of [7], namely $m = 0$, $n = 1$, $R_{01}^{(1)}$ is odd in ξ and $R_{01}^{(2)}$ is even in ξ. Further $R_{01}^{(2)}(\beta, \xi)$, shown for $\beta = 2$ in figure 2 together with its derivative, is not analytic at $\xi = 0$, having a discontinuous derivative there. This is the cause of the blade-like appearance of the $|\psi|^2$ plot (figure 3, [7]), and also means that the radial equation (6) is not satisfied inside the disc $\rho \leqslant b$, $z = 0$.

We now demonstrate a more general result, not dependent on specific properties of the $R_{mn}^{(1,2)}(\beta, \xi)$ and $S_{mn}(\beta, \eta)$

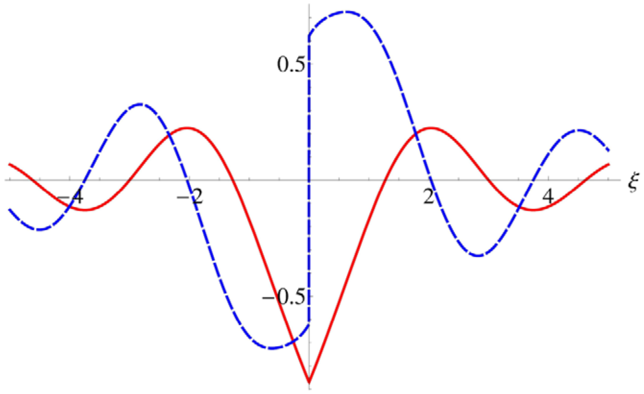

Figure 2. The radial spheroidal function $R_{01}^{(2)}(\beta, \xi)$ and its derivative (solid and dashed lines, respectively), drawn for the highly non-paraxial case $\beta = 2$. As β increases, the discontinuity in the derivative at $\xi = 0$ rapidly decreases, and is not visible on this scale when $\beta = 5$.

spheroidal functions, namely that no single separable product wavefunction of the form

$$\psi(\xi, \eta, \phi) = R(\xi)S(\eta)\Phi(\phi) \quad (15)$$

can represent a scalar beam. From (11) we have

$$\partial_z \psi|_{\xi=0} = \frac{1}{b\eta} R'(0)S(\eta)\Phi(\phi) \quad (16)$$

$$\partial_z \psi|_{\eta=0} = \frac{1}{b\xi} R(\xi)S'(0)\Phi(\phi). \quad (17)$$

The derivative $\partial_z \psi$ must be finite on the critical circle $\{\rho = b, z = 0\}$ or $\{\xi = 0, \eta = 0\}$ (condition (iii)). Thus the necessary (but not sufficient) conditions for the finiteness of $\partial_z \psi$ on the critical circle are

$$R'(0)S(0) = 0 \qquad \text{and} \qquad R(0)S'(0) = 0. \quad (18)$$

The conditions (18) will be satisfied if one or more of the following hold:

$$\begin{array}{llll} \text{(a)} & R(0) = 0 & \text{and} & S(0) = 0 \\ \text{(b)} & R'(0) = 0 & \text{and} & S'(0) = 0 \\ \text{(c)} & R(0) = 0 & \text{and} & R'(0) = 0 \\ \text{(d)} & S(0) = 0 & \text{and} & S'(0) = 0. \end{array} \quad (19)$$

If (a) holds then $\psi(\rho, z = 0) = 0$, making J_z zero in the entire focal plane, which is contrary to condition (iii).

If (b) holds then $\partial_z \psi|_{z=0} = 0$, again making J_z zero in the entire focal plane.

If (c) holds then $\psi(\rho \leqslant b, z = 0) = 0$, giving zero J_z through the central disc of the focal plane, which is non-physical (consider paraxial beams).

If (d) holds then both ψ and $\partial_z \psi$ will be zero for $\eta = 0$, i.e. for $\{\rho \geqslant b, z = 0\}$, not a fatal flaw. But if $S(0)$ and $S'(0)$ were both zero, the series expansion of $S(\eta)$ would start with the second or higher power of η, whereas the indicial equation of (7) gives the exponents 0 and 1.

Thus the separable form of (15) cannot represent a scalar beam.

Electromagnetic beams may be constructed from scalar solutions of the wave equation. For example [13, 14] the TM, TE, 'LP', and 'CP' beams have their vector potential **A** proportional (respectively) to

$$(0, 0, \psi), \quad (\partial_y \psi, -\partial_x \psi, 0), \quad (\psi, 0, 0) \quad \text{and} \quad (-\mathrm{i}\psi, \psi, 0). \quad (20)$$

(The quotation marks indicate that the 'LP' and 'CP' beams are fully linearly and circularly polarized only in the plane wave limit: it has been shown that perfect linear or circular polarization cannot exist in finite beams [14].)

The expressions in (20) are only the simplest of an infinity of possible vector potentials representing electromagnetic beams. As these vector potentials depend linearly on ψ or its derivatives, the reasons given against the form $\psi = R(\xi)S(\eta)\,\mathrm{e}^{\mathrm{i}m\phi}$ for scalar beams make it unlikely, in our view, that a separable wavefunction can be made the basis of physical electromagnetic beams.

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