Brewster angles in reflection by uniaxial crystals

John Lekner

Department of Physics, Victoria University of Wellington, Wellington, New Zealand

Received July 13, 1992; revised manuscript received April 1, 1993; accepted April 2, 1993

Reflection by anisotropic media is characterized by the four reflection amplitudes r_{pp}, r_{ss}, r_{ps}, and r_{sp}. We show that r_{pp} can be zero at angles θ_{pp}, the anisotropic Brewster angles, and that a quantity related to θ_{pp} satisfies a quartic equation. When the refractive index of the medium of incidence lies between the ordinary and the extraordinary indices of the crystal, it is possible for r_{ss} to be zero at an angle θ_{ss}, and there exist four equivalent orientations of the crystal optic axis for which r_{pp}, r_{ss}, and either r_{ps} or r_{sp} are simultaneously zero, at angle of incidence equal to arctan (n_o/n_1).

1. INTRODUCTION

It is well known[1-3] that a p-polarized wave has zero reflection from an isotropic material at the Brewster angle θ_B given by tan $\theta_B = n/n_1$, where n is the refractive index of the (nonabsorbing) material and n_1 is the refractive index of the medium of incidence. Special cases of the analogous zero reflection of p-polarized incident waves into the p polarization by uniaxial crystals are known[2-4]; here we will examine the general case of arbitrary orientation of the optic axis relative to the reflecting plane and to the plane of incidence. We find that it is possible to find the angles θ_{pp}, at which the r_{pp} reflection amplitude is zero, by solving a quartic equation. Surprisingly, the r_{ss} reflection amplitude also can be zero when the refractive index of the medium of incidence lies between the ordinary and the extraordinary indices. The angle θ_{ss} at which this occurs is found by solving a quartic equation, which is related to the quartic equation that determines θ_{pp}. Further, for particular orientation of the crystal optic axis and at an angle of incidence whose tangent equals the ratio of the ordinary index to the index of the medium of incidence, r_{pp}, r_{ss}, and one of r_{ps} or r_{sp} can be zero together.

2. REFLECTION BY A UNIAXIAL CRYSTAL

I will use the results of a recent paper[5] on the optical properties of uniaxial crystals. Let θ be the angle of incidence at which an incident plane wave strikes a planar surface of a uniaxial crystal. The z axis is defined as the inward normal to the reflecting surface, which lies in the x–y plane. The x axis lies in the plane of incidence, the y axis normal to it. Then the x components of the incident, the ordinary, and the extraordinary wave vectors $\mathbf{k}_1$, $\mathbf{k}_o$ and $\mathbf{k}_e$ are identical, with value

$$K = k_1 \sin\theta = n_1 \frac{\omega}{c} \sin\theta. \tag{1}$$

The z components of the incident and the reflected waves are $\pm q_1$, where

$$q_1^2 = k_1^2 - K^2 = k_1^2 \cos^2\theta. \tag{2}$$

The wave vector $\mathbf{k}_o$ of the ordinary wave inside the crystal has magnitude

$$k_o = n_o \frac{\omega}{c}, \tag{3}$$

and normal component q_o given by

$$q_o^2 = k_o^2 - K^2. \tag{4}$$

The extraordinary wave vector has components $(K, 0, q_e)$, where

$$q_e = \bar{q} - \alpha\gamma K \Delta\epsilon/\epsilon_\gamma, \tag{5}$$

where $\Delta\epsilon = \epsilon_e - \epsilon_o \equiv n_e^2 - n_o^2$, the optic axis has direction cosines α, β, and γ relative to the x, y, and z axes, respectively, and

$$\bar{q}^2 = \epsilon_o[\epsilon_e\epsilon_\gamma\omega^2/c^2 - K^2(\epsilon_e - \beta^2\Delta\epsilon)]/\epsilon_\gamma^2, \tag{6}$$

with $\epsilon_\gamma = n_\gamma^2$ defined as

$$\epsilon_\gamma = \epsilon_o + \gamma^2\Delta\epsilon. \tag{7}$$

The reflection amplitudes found in Ref. 5 may be written in the form

$$r_{ss} = \frac{a(q_1 - q_o) + b(q_1 - q_e)}{a(q_1 + q_o) + b(q_1 + q_e)}, \tag{8}$$

$$-r_{pp} = \frac{a'(q_1 + q_o) + b'(q_1 + q_e)}{a(q_1 + q_o) + b(q_1 + q_e)}, \tag{9}$$

$$r_{sp,ps} = \frac{2\beta(\alpha q_o \pm \gamma K)(q_o - q_e)k_1 k_o^2}{a(q_1 + q_o) + b(q_1 + q_e)} \tag{10}$$

(the plus corresponds to r_{sp}, the minus to r_{ps}), where

$$\begin{aligned} a &= (\alpha q_o - \gamma K)[\alpha(k_o^2 q_e + q_t q_o^2) - \gamma K(k_o^2 + q_t q_e)], \\ a' &= (\alpha q_o - \gamma K)[\alpha(k_o^2 q_e - q_t q_o^2) - \gamma K(k_o^2 - q_t q_e)], \\ b &= \beta^2 k_o^2(k_o^2 + q_t q_o), \qquad b' = \beta^2 k_o^2(k_o^2 - q_t q_o), \end{aligned} \tag{11}$$

and $q_t = k_1^2/q_1$. We note that the reflection amplitudes are particularly simple when $\beta = 0$ (that is, when the optic axis lies in the plane of incidence); then all b, b', r_{sp}, and

0740-3232/93/092059-06$06.00

r_{ps} are zero, and r_{ss} and r_{pp} simplify to

$$r_{ss}(\beta = 0) = \frac{q_1 - q_o}{q_1 + q_o}, \qquad r_{pp}(\beta = 0) = \frac{Q - Q_1}{Q + Q_1}, \tag{12}$$

with $Q_1 = q_1/\epsilon_1$, $Q = q_\gamma/n_o n_e$, where

$$q_\gamma^2 = \epsilon_\gamma \omega^2/c^2 - K^2. \tag{13}$$

In this case the Brewster angle (at which r_{pp} is zero) is found by solving $Q = Q_1$, which gives

$$\tan^2 \theta_{pp}(\beta = 0) = \frac{\epsilon_o\epsilon_e - \epsilon_1\epsilon_\gamma}{\epsilon_1(\epsilon_\gamma - \epsilon_1)}. \tag{14}$$

This formula contains as special cases two known results[4,5] for the Brewster angle when the optic axis coincides with the x and the z axes:

$$\tan^2 \theta_{pp}(\alpha^2 = 1) = \frac{\epsilon_o(\epsilon_e - \epsilon_1)}{\epsilon_1(\epsilon_o - \epsilon_1)}, \tag{15}$$

$$\tan^2 \theta_{pp}(\gamma^2 = 1) = \frac{\epsilon_e(\epsilon_o - \epsilon_1)}{\epsilon_1(\epsilon_e - \epsilon_1)}. \tag{16}$$

These two expressions give bounds on the Brewster angle.[6]

3. DETERMINATION OF θ_{pp}

The Brewster angle θ_{pp} is that angle at which the numerator of Eq. (9) is zero. This numerator contains the angle of incidence in the variable K, which appears linearly in q_e and α' and also inside a square root in q_e, q_1, q_o, and q_t. An algebraic equation can be obtained by use of a symmetry of r_{pp} and by introducing a variable related to K^2 through a bilinear transformation.

The reflection amplitudes r_{pp} and r_{ss} are unchanged by change of sign of any of the three optic axis direction cosines α, β, and γ. Reversal of the sign of any one of the direction cosines of the optic axis is equivalent to reversal of the appropriate x, y, or z axis. It is clear from Eqs. (5) and (8)–(11) that r_{ss} and r_{pp} can depend only on the sign of $\alpha\gamma$, and the invariance of r_{ss} and r_{pp} to the sign of $\alpha\gamma$ follows from the algebraic identity

$$\bar{q}^2 = q_o^2 + [\alpha^2 q_o^2 + \beta^2 k_o^2 + \gamma^2 K^2 + (\alpha\gamma)^2 K^2 \Delta\epsilon/\epsilon_\gamma]\Delta\epsilon/\epsilon_\gamma. \tag{16a}$$

The numerators and the denominators of r_{pp} and r_{ss} as given by Eqs. (8) and (9) can be split up into invariant (I) and variant (V) parts, namely those that are even and odd in $\alpha\gamma$. Both r_{pp} and r_{ss} are thus of the form

$$\frac{I' + V'}{I + V} = \left(\frac{1 + V'/I'}{1 + V/I}\right)\frac{I'}{I} = \left(\frac{I'/V' + 1}{I/V + 1}\right)\frac{V'}{V}. \tag{17}$$

For an expression of this type to be invariant we must have $V'/I' = V/I$, and the value of Eq. (17) is thus $I'/I = V'/V$. When the latter form is used, the r_{ss} and r_{pp} reflection amplitudes reduce to

$$r_{ss} = \frac{(q_1 - q_o)[(q_o + \bar{q})(k_o^2 + q_t q_o) + (\alpha^2 k_o^2 q_o + \gamma^2 K^2 q_t)\Delta\epsilon/\epsilon_\gamma] - \beta^2 k_o^2 (k_o^2 + q_t q_o)\Delta\epsilon/\epsilon_\gamma}{(q_1 + q_o)[(q_o + \bar{q})(k_o^2 + q_t q_o) + (\alpha^2 k_o^2 q_o + \gamma^2 K^2 q_t)\Delta\epsilon/\epsilon_\gamma] + \beta^2 k_o^2 (k_o^2 + q_t q_o)\Delta\epsilon/\epsilon_\gamma},$$

$$-r_{pp} = \frac{(q_1 - q_o)[(q_o + \bar{q})(k_o^2 - q_t q_o) + (\alpha^2 k_o^2 q_o - \gamma^2 K^2 q_t)\Delta\epsilon/\epsilon_\gamma] + \beta^2 k_o^2 (k_o^2 - q_t q_o)\Delta\epsilon/\epsilon_\gamma}{(q_1 + q_o)[(q_o + \bar{q})(k_o^2 + q_t q_o) + (\alpha^2 k_o^2 q_o + \gamma^2 K^2 q_t)\Delta\epsilon/\epsilon_\gamma] + \beta^2 k_o^2 (k_o^2 + q_t q_o)\Delta\epsilon/\epsilon_\gamma}. \tag{18}$$

These expressions are manifestly invariant to changes in the sign of the direction cosines. The angle of incidence now appears only through $K^2 = k_1^2 \sin^2 \theta$, but K^2 is inside a square root in the variables q_1, q_t, q_o, and $\bar{q}$. Consider the effect of changing to the variable ϵ defined by

$$\epsilon\omega^2/c^2 = q_t q_o \qquad \text{or} \qquad \epsilon/\epsilon_1 = q_o/q_1. \tag{19}$$

The variables dependent on the angle of incidence may be written in terms of ϵ:

$$\left(\frac{cK}{\omega}\right)^2 = \frac{\epsilon_1(\epsilon^2 - \epsilon_1\epsilon_o)}{\epsilon^2 - \epsilon_1^2}, \qquad \left(\frac{cq_1}{\omega}\right)^2 = \frac{\epsilon_1^2(\epsilon_o - \epsilon_1)}{\epsilon^2 - \epsilon_1^2},$$
$$\left(\frac{cq_o}{\omega}\right)^2 = \frac{\epsilon^2(\epsilon_o - \epsilon_1)}{\epsilon^2 - \epsilon_1^2}. \tag{20}$$

Note that $q_1 q_o$ expressed in terms of ϵ does not involve a square root. The numerator of r_{pp} now simplifies to $(\epsilon + \epsilon_1)\omega^4/c^4$ times

$$(\epsilon_o - \epsilon)(\epsilon - \epsilon_1)(q_1 + q_o)\bar{q}c^2/\omega^2 + \epsilon(\epsilon_o - \epsilon)(\epsilon_o - \epsilon_1)$$
$$+ [\alpha^2\epsilon_o\epsilon(\epsilon_o - \epsilon_1) + \beta^2\epsilon_o(\epsilon_o - \epsilon)(\epsilon - \epsilon_1)$$
$$- \gamma^2\epsilon_1(\epsilon^2 - \epsilon_1\epsilon_o)]\Delta\epsilon/\epsilon_\gamma. \tag{21}$$

The remaining square roots are all in the first term. To find the equation determining θ_{pp} we set expression (21) equal to zero, isolate the first term, square, and use Eqs. (6), (20), and

$$(q_1 + q_o)^2 = \frac{(\epsilon_1 + \epsilon)(\epsilon_o - \epsilon_1)}{\epsilon - \epsilon_1}\frac{\omega^2}{c^2}. \tag{22}$$

The resulting equation is a quartic in ϵ, namely,

$$a_0 + a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + a_4\epsilon^4 = 0,$$
$$a_0 = -\epsilon_1^2\epsilon_o\{\epsilon_o^2(\epsilon_o - \epsilon_1)(\beta^2 + \gamma^2)$$
$$+ [(\beta^2\epsilon_o - \gamma^2\epsilon_1)^2 + \epsilon_o(\epsilon_o - \epsilon_1)\gamma^2]\Delta\epsilon\},$$
$$a_1 = 2\beta^2\epsilon_1\epsilon_o(\epsilon_o(\epsilon_o^2 - \epsilon_1^2)$$
$$+ \{\epsilon_o[\epsilon_o + \epsilon_1(2\beta^2 - 1)] - 2\gamma^2\epsilon_1^2\}\Delta\epsilon),$$
$$a_2 = (\epsilon_o - \epsilon_1)\epsilon_o[\epsilon_1\epsilon_o(1 - 5\beta^2) + \epsilon_1^2(\gamma^2 - \beta^2)$$
$$- \epsilon_o^2(1 - \gamma^2)] + ((\epsilon_o - \epsilon_1)^2(\epsilon_o + \epsilon_1)\gamma^2$$
$$- \epsilon_o\{2\epsilon_1(\beta^2\epsilon_o - \gamma^2\epsilon_1)(\beta^2 + \gamma^2)$$
$$+ [\epsilon_o + \epsilon_1(2\beta^2 - 1)]^2\})\Delta\epsilon,$$
$$a_3 = 2\beta^2\epsilon_o\{\epsilon_o^2 - \epsilon_1^2 + [\epsilon_1(2\beta^2 + 2\gamma^2 - 1) + \epsilon_o]\Delta\epsilon\},$$
$$a_4 = (\epsilon_o - \epsilon_1)[\epsilon_o(1 - 2\beta^2 - \gamma^2) - \epsilon_1(1 - \beta^2)]$$
$$+ [(\epsilon_o - \epsilon_1)\gamma^2 - \epsilon_o(\beta^2 + \gamma^2)^2]\Delta\epsilon. \tag{23}$$

The coefficients a_1 and a_3 are zero when the optic axis lies in the plane of incidence ($\beta = 0$). The quartic then reduces to a quadratic in ϵ^2, with roots

$$\frac{\gamma^2\epsilon_1\epsilon_o}{\epsilon_1 - (1 - \gamma^2)\epsilon_o}, \qquad \frac{\epsilon_o\epsilon_e(\epsilon_o - \epsilon_1) + \epsilon_1^2(\epsilon_\gamma - \epsilon_o)}{\epsilon_\gamma - \epsilon_1}. \tag{24}$$

Only the latter value of ϵ^2 is a physical root: the other value does not make expression (21) zero. Since, from Eqs. (20),

$$\tan^2\theta = \frac{\epsilon^2 - \epsilon_1\epsilon_o}{\epsilon_1(\epsilon_o - \epsilon_1)}, \tag{25}$$

the second value of ϵ^2 in expression (24) reproduces Eq. (14) and thus also the $\alpha^2 = 1$ and $\gamma^2 = 1$ θ_{pp} expressions given in Eqs. (15) and (16). When $\beta^2 = 1$, the roots of Eq. (23) are ϵ_1, ϵ_1, ϵ_o, and ϵ_o. Again $\epsilon = \epsilon_1$ does not make expression (21) zero. The (double) physical root is ϵ_o, and for this root Eq. (25) gives

$$\tan^2\theta_{pp}(\beta^2 = 1) = \frac{\epsilon_o}{\epsilon_1}. \tag{26}$$

(This value also holds when $\alpha^2\epsilon_o = \gamma^2\epsilon_1$, for any β.) Thus when the optic axis is normal to the plane of incidence, the zero of r_{pp} occurs at the Brewster value for an isotropic medium of refractive index $n_o, \theta_B = \arctan(n_o/n_1)$, as is known.[4,5]

An algorithm for the algebraic solution of a general quartic equation is known,[7] and so it is possible in principle to give an analytic general solution of expression (21). I have been unable to simplify this algebraic solution to a form short enough to be useful or to find factors of expression (21) (which would allow the degree of the equation to be reduced). It is possible, however, to give a simple form for the solution to first order in the anisotropy $\Delta\epsilon = \epsilon_e - \epsilon_o$ by expanding about $\epsilon = \epsilon_o$, which is the physical root of expression (21) in the isotropic case. The result is

$$\epsilon \approx \epsilon_o + (\alpha^2\epsilon_o - \gamma^2\epsilon_1)\Delta\epsilon/2\epsilon_o. \tag{27}$$

The resulting value of $\tan^2\theta_{pp}$ as given in Eq. (25), to first order in $\Delta\epsilon$, is

$$\tan^2\theta_{pp} \approx \frac{\epsilon_o}{\epsilon_1} - \frac{(\alpha^2\epsilon_o - \gamma^2\epsilon_1)\Delta\epsilon}{\epsilon_1(\epsilon_o - \epsilon_1)}. \tag{28}$$

This approximation gives the exact Brewster angle when $\alpha^2 = 1$ (optic axis coincident with the intersection of the plane of incidence and the reflecting surface), when $\beta^2 = 1$ (optic axis perpendicular to the plane of incidence) and also when $\alpha^2\epsilon_o = \gamma^2\epsilon_1$, for any β. Expression (28) is least accurate when $\gamma^2 = 1$ (optic axis normal to the reflecting surface); for calcite in air expression (28) then gives $\theta_{pp} \approx 60.19°$, whereas the exact value from Eq. (16) is 60.72°, an error of 0.9%. Figure 1 shows the exact θ_{pp} [Eq. (14)] and the approximate Brewster angles of expression (28) when the optic axis lies in the plane of incidence, for calcite in air.

A better approximation to the Brewster angle is one that contains the $\beta = 0$, $\beta^2 = 1$, and $\alpha^2\epsilon_o = \gamma^2\epsilon_1$ known solutions (14) and (26) as special cases, namely,

$$\tan^2\theta_{pp} \approx \frac{\epsilon_o\epsilon_{\alpha\gamma} - \epsilon_1\epsilon_\gamma}{\epsilon_1(\epsilon_\gamma - \epsilon_1)}, \tag{29}$$

where

$$\epsilon_{\alpha\gamma} = \epsilon_o + (\alpha^2 + \gamma^2)\Delta\epsilon = \epsilon_e - \beta^2\Delta\epsilon. \tag{30}$$

This expression for the Brewster angle would show zero error in Fig. 1, and for calcite in air it has θ_{pp} errors between −0.2% and +0.4%.

4. EFFECT OF INDEX MATCHING

The reader may have noted that the bounds on the Brewster angle θ_{pp} given by Eqs. (15) and (16) tend to 0° and 90°, respectively, as ϵ_1 increases toward the smaller of ϵ_o and ϵ_e. Figure 2 shows the Brewster angle for calcite immersed in a liquid of refractive index 1.48, when the optic axis lies in the plane of incidence. The Brewster angle now ranges from 10.4° to 80.7°, and the error in θ_{pp} as given by expanding to first order in the anisotropy $\Delta\epsilon$, namely expression (28), now ranges to nearly −31%. [The Brewster angle given by expression (29), exact for $\beta = 0$ and $\beta^2 = 1$, now has errors between −6% and +5%.]

The effect of index matching is thus to enhance the anisotropy, as could already be seen in expression (28), where $\Delta\epsilon$ is divided by $\epsilon_o - \epsilon_1$. Similar enhancement of anisotropy by index matching has been calculated for optical properties of a uniaxial substrate covered by an isotropic

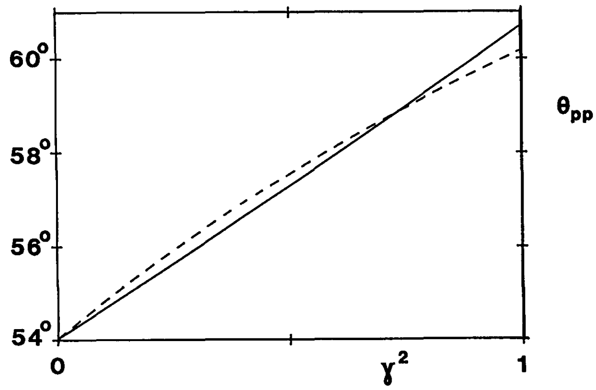

Fig. 1. Exact and approximate Brewster angles for calcite in air at 633 nm ($n_o = 1.655$, $n_e = 1.485$). The curves show θ_{pp} for $\beta = 0$ (optic axis in the plane of incidence) as a function of γ^2, the square of the cosine of the angle between the optic axis and the normal to the reflecting surface. The solid curve is from Eq. (14), the dashed curve from expression (28).

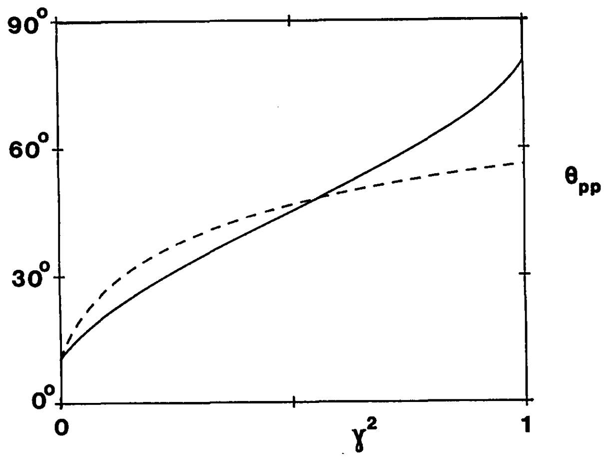

Fig. 2. Brewster angles for calcite with optic axis in the plane of incidence, immersed in a medium of refractive index 1.48, as a function of γ^2. The notation is as for Fig. 1, but note the change of vertical scale. The curves cross at $\gamma^2 = \epsilon_o/(\epsilon_1 + \epsilon_o)$.

layer[8] (for example, the effect of the anisotropy of ice is enhanced a hundredfold by the presence of a water layer).

What happens when the index of the ambient medium lies between the ordinary and the extraordinary indices? Then $\epsilon_o - \epsilon_1$ and $\epsilon_e - \epsilon_1$ have opposite signs, and neither Eq. (15) nor Eq. (16) gives a real Brewster angle. From Eq. (14) we see that, for a zero of r_{pp} to be possible when the optic axis lies in the plane of incidence, $\epsilon_\gamma = \epsilon_o + \gamma^2\Delta\epsilon$ must lie between ϵ_1 and $\epsilon_o\epsilon_e/\epsilon_1$. This range is largest when ϵ_1 is close to ϵ_o or ϵ_e, and it shrinks to zero when ϵ_1 tends to the geometric mean of ϵ_o and ϵ_e, i.e., to n_on_e. When ϵ_1 is close to n_on_e there is extreme sensitivity to the orientation of the optic axis relative to the normal to the reflecting plane, since for a very small range of γ^2 expression (14) then gives values of θ_{pp} ranging from 0° to 90°. Of these, only those smaller than the appropriate critical angle θ_c^o or θ_c^e are possible, since when either q_o or $\overline{q}$ is imaginary the r_{pp} reflection amplitude is complex, and expressions (14), (15), and (16) no longer apply. The critical angles θ_c^o and θ_c^e are here defined as those angles that make q_o and $\overline{q}$ zero; from Eqs. (4) and (6) we thus have

$$\sin\theta_c^o = \frac{n_o}{n_1}, \qquad \sin\theta_c^e = \frac{n_e n_\gamma}{n_1 n_{\alpha\gamma}}, \tag{31}$$

where $n_{\alpha\gamma}$ is the square root of $\epsilon_{\alpha\gamma} = \epsilon_o + (\alpha^2 + \gamma^2)\Delta\epsilon$. When the optic axis lies in the plane of incidence, $\beta = 0$ and $\alpha^2 + \gamma^2 = 1$, so that

$$\sin\theta_c^e(\beta = 0) = \frac{n_\gamma}{n_1}. \tag{32}$$

From Eq. (14) we have that

$$\sin^2\theta_{pp}(\beta = 0) = \frac{\epsilon_o\epsilon_e - \epsilon_1\epsilon_\gamma}{\epsilon_o\epsilon_e - \epsilon_1^2}. \tag{33}$$

Since, as noted above, ϵ_γ must lie between ϵ_1 and $\epsilon_o\epsilon_e/\epsilon_1$ for real θ_{pp}, the condition $\theta_{pp}(\beta = 0) \le \theta_c^e(\beta = 0)$ holds for all γ for which Eq. (14) gives a real Brewster angle. For positive anisotropy ($\Delta\epsilon = \epsilon_e - \epsilon_o > 0$), the condition $\theta_{pp} < \theta_c^o$ is satisfied for

$$\gamma^2 > \frac{\epsilon_o\epsilon_e(\epsilon_1 - \epsilon_o)}{\epsilon_1^2(\epsilon_e - \epsilon_o)} = \gamma_c^2 \tag{34}$$

when $\epsilon_1 < n_on_e$, and for $\gamma^2 < \gamma_c^2$ when $\epsilon_1 > n_on_e$. Thus when $\epsilon_e < \epsilon_1 < \epsilon_o$, negative uniaxial crystals have θ_{pp} ranging from 0 to 90°, while positive uniaxial crystals immersed in a medium such that $\epsilon_o < \epsilon_1 < \epsilon_e$ have θ_{pp} ranging from 0 to θ_c^o. In either case the $\beta = 0$ Brewster angle exists only for a range of inclinations of the optic axis to the normal, since the square of the direction cosine γ must lie between

$$\left|\frac{\epsilon_o - \epsilon_1}{\epsilon_o - \epsilon_e}\right| \quad\text{and}\quad \frac{\epsilon_o}{\epsilon_1}\left|\frac{\epsilon_1 - \epsilon_e}{\epsilon_o - \epsilon_e}\right|. \tag{35}$$

Figure 3 shows $\theta_{pp}(\beta = 0)$ for calcite immersed in a liquid of refractive index 1.51.

The fact that index matching can extend the range of the Brewster angle from normal to grazing incidence has interesting implications. At grazing incidence $q_1 \to 0$, and we see from Eqs. (9) and (11) that $r_{pp} \to 1$. Thus if θ_{pp} is close to 90° there is a conflict of limits, and we may expect a very rapid variation in the p to p reflectivity, from zero to unity in a small range near grazing incidence.

One other unexpected phenomenon associated with index matching is the occurrence of two zeros of r_{pp}, at different angles of incidence, for some crystal orientations. The double Brewster angles are associated with q_e becoming complex for $\theta > \theta_c^e$. [At θ_c^e, $\overline{q}$ is zero, and for greater angles of incidence $\overline{q}$ is imaginary; θ_c^e is given by Eq. (31).] Figure 4 shows r_{pp} for calcite immersed in a medium of refractive index 1.51, with the squares of the direction cosines of the optic axis $\alpha^2 = 0.02$, $\beta^2 = 0.67$, $\gamma^2 = 0.31$. Beyond θ_c^e the reflection amplitude r_{pp} is complex, and both the real and the imaginary parts are shown. Note that the second Brewster angle is just a bit less then θ_c^e and that $|r_{pp}|^2$ is not unity for $\theta > \theta_c^e$ except at grazing incidence.

5. ANGLES AT WHICH $r_{ss} = 0$

An isotropic medium has zero r_s only when perfectly index matched, in which case r_s is zero for all angles of incidence.

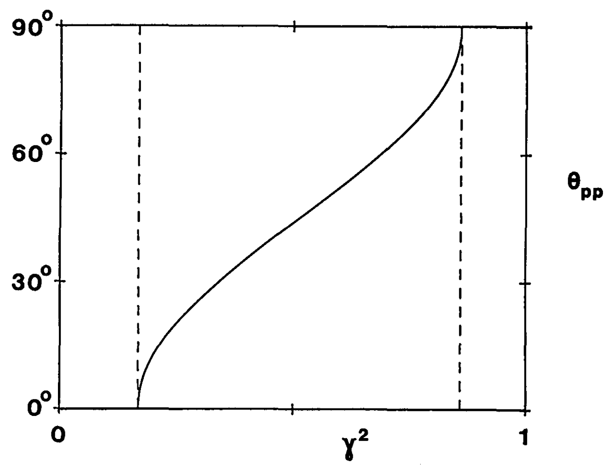

Fig. 3. Brewster angles for calcite immersed in a medium of index $n_1 = 1.51$, for the optic axis in the plane of incidence. Note that a zero of r_{pp} exists only for a range of inclinations of the optic axis to the surface normal (here between 22° and 66°) as given by expressions (35). These bounding values of the square of the direction cosine γ are shown by vertical dashed lines.

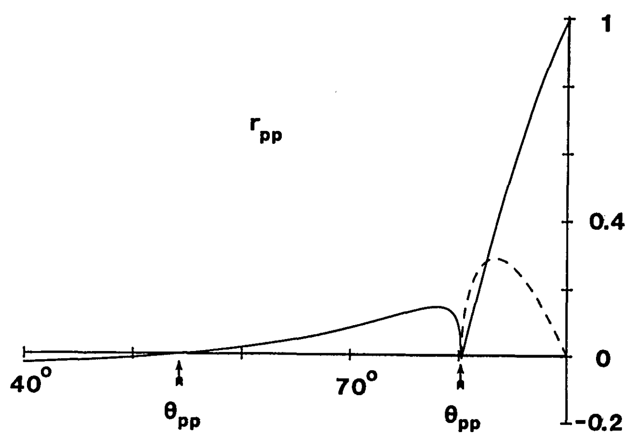

Fig. 4. Reflection amplitude r_{pp} for calcite immersed in a liquid of refractive index 1.51 and direction cosines of the optic axis given by $\alpha^2 = 0.02, \beta^2 = 0.67, \gamma^2 = 0.31$. Beyond θ_c^e there is an imaginary part of r_{pp} (dashed curve). The θ_{pp} values are 54.03° and 80.27°; the value of θ_c^e is just 1 mdeg more than the larger θ_{pp}.

We now examine the possibility of zeros of r_{ss}, starting from formulas (18). The procedure and the variables that we use are as for the zeros of r_{pp}, so we need state only the final result, which is that when r_{ss} is zero a quartic equation in $\epsilon = \epsilon_1 q_o/q_1$ is satisfied. This quartic is almost identical to the r_{pp} quartic [Eq. (23)], the difference being that the signs of the coefficients a_1 and a_3 are reversed. Now a_1 and a_3 are proportional to β^2, and thus solutions of the r_{pp} and r_{ss} quartics are the same when β is zero. It is *not* true, however, that r_{ss} is zero at the angle given by Eq. (14) when $\beta = 0$. This is not a physical root, and in fact when $\beta = 0$, $r_{ss} = (q_1 - q_o)/(q_1 + q_o)$, which is not zero at any angle unless $n_o = n_1$, in which case it is zero at all angles of incidence. When $\beta^2 = 1$ the r_{ss} quartic factors to $(\epsilon + \epsilon_1)^2(\epsilon + \epsilon_o)^2(\epsilon_e - \epsilon_1)$ times a constant, and thus r_{ss} can be zero only when $\epsilon_1 = \epsilon_e$ for $\beta^2 = 1$, since $\epsilon = \epsilon_1 q_o/q_1$ must be positive. The $\alpha^2 = 1$ and $\gamma^2 = 1$ roots are simple but also nonphysical, being a subset of the $\beta = 0$ roots.

When $\alpha^2\epsilon_o = \gamma^2\epsilon_1$, the quartic factors to $(\epsilon + \epsilon_o)^2$ times the quadratic

$$\begin{aligned} &a\epsilon^2 + b\epsilon + c = 0: \\ a &= (\epsilon_o^2 - \epsilon_1^2)[\epsilon_1^2 + \beta^2(\epsilon_o^2 + \epsilon_1\epsilon_o - \epsilon_1^2)] \\ &\quad + \epsilon_o[\epsilon_1^2 + \beta^2(\epsilon_o^2 + 2\epsilon_1\epsilon_o - \epsilon_1^2 + \epsilon_1^2\beta^2)]\Delta\epsilon, \\ b &= 2\epsilon_1\epsilon_o[\epsilon_1(2\beta^2 - 1) + \beta^2\epsilon_o][\epsilon_o^2 - \epsilon_1^2 + (\epsilon_o + \beta^2\epsilon_1)\Delta\epsilon], \\ c &= \epsilon_o\epsilon_1^2\{(\epsilon_o^2 - \epsilon_1^2)(\epsilon_o + \beta^2\epsilon_1) + [\epsilon_o^2(1 - \beta^2 + \beta^4) \\ &\quad + 2\epsilon_1\epsilon_o\beta^2(2\beta^2 - 1) + \beta^2\epsilon_1^2(4\beta^2 - 3)]\Delta\epsilon\}. \end{aligned} \tag{36}$$

The discriminant $b^2 - 4ac$ is zero when $\beta^2 = \beta_d^2$, where

$$\beta_d^2 = \frac{(\epsilon_o\epsilon_e - \epsilon_1^2)[(\epsilon_o + \epsilon_1)^2 + (3\epsilon_1 + \epsilon_o)\Delta\epsilon]}{\Delta\epsilon[\epsilon_o^3 + 2\epsilon_o^2\epsilon_1 - 3\epsilon_o\epsilon_1^2 - 4\epsilon_1^3 + \epsilon_o(3\epsilon_1 + \epsilon_o)\Delta\epsilon]}. \tag{37}$$

I have given the details of the $\alpha^2\epsilon_o = \gamma^2\epsilon_1$ solution because it is the only simple root of the $r_{ss} = 0$ quartic known to me and because this configuration will be seen (in Section 6) to give the possibility of simultaneous zeros r_{ss} and r_{pp}.

Figure 5 shows r_{ss} as a function of the angle of incidence for the same system that was used to illustrate the existence of double Brewster angles in Fig. 4.

6. SIMULTANEOUS ZERO OF r_{pp} AND r_{ss}

Since r_{ss} as well as r_{pp} can be zero when the refractive index of the medium of incidence lies between the ordinary and the extraordinary indices of the crystal, it is natural to ask whether r_{ss} and r_{pp} can be zero at the same angle of incidence. In this section I show that they can (under certain conditions) and that when this happens, one of r_{sp} or r_{ps} will also be zero.

We equate the numerators of r_{ss} and r_{pp} in Eqs. (18) to zero. From these two simultaneous equations, linear in $\bar{q}$, we can eliminate $\bar{q}$ and solve for $\epsilon = \epsilon_1 q_o/q_1$. We find that ϵ is determined by an equation linear in ϵ^2 and independent of $\Delta\epsilon$, the solution of which is

$$\epsilon^2 = \frac{\epsilon_1\epsilon_o(\beta^2\epsilon_o - \gamma^2\epsilon_1)}{\epsilon_1(2\beta^2 - 1) + \epsilon_o\alpha^2}. \tag{38}$$

Both the $r_{pp} = 0$ and the $r_{ss} = 0$ quartics must be satisfied by this value. Thus we must have, for ϵ^2 as given in Eq. (38),

$$(a_0 + a_2\epsilon^2 + a_4\epsilon^4)^2 = (a_1 + a_3\epsilon^2)\epsilon^2. \tag{39}$$

This condition is satisfied by $\epsilon_o = \epsilon_1$ and by

$$\alpha^2\epsilon_o = \gamma^2\epsilon_1. \tag{40}$$

The latter is the physical root, leading to $\epsilon = \epsilon_o$ and thus from Eq. (25) to

$$\tan^2\theta_j = \frac{\epsilon_o}{\epsilon_1}. \tag{41}$$

Thus the joint zero of r_{pp} and r_{ss} can occur at one angle of incidence, which is the same as the Brewster angle for an isotropic medium of refractive index n_o, namely, $\arctan(n_o/n_1)$.

The special crystal orientations for which this joint zero can occur follow on substitution of Eq. (40) and its consequence,

$$\gamma^2 = (1 - \beta^2)\frac{\epsilon_o}{\epsilon_1 + \epsilon_o}, \tag{42}$$

into the $r_{ss} = 0$ quartic. This leads to a linear equation in β^2 and thus to the values

$$\begin{aligned} \alpha_j^2 &= (\epsilon_1 + \epsilon_o)(\epsilon_e - \epsilon_1)/4\epsilon_1\Delta\epsilon, \\ \beta_j^2 &= (\epsilon_1 - \epsilon_o)[(\epsilon_1 + \epsilon_o)^2 + (3\epsilon_1 + \epsilon_o)\Delta\epsilon]/4\epsilon_1^2\Delta\epsilon, \\ \gamma_j^2 &= \epsilon_o(\epsilon_1 + \epsilon_o)(\epsilon_e - \epsilon_1)/4\epsilon_1^2\Delta\epsilon. \end{aligned} \tag{43}$$

Thus for crystal orientations $(\pm|\alpha_j|, \pm|\beta_j|, \pm|\gamma_j|)$ and at angle of incidence $\theta_j = \arctan(n_o/n_1)$, r_{ss} and r_{pp} will be zero together. For such orientations to exist, each of the squares of the direction cosines given in Eq. (42) must be positive. This will be so if ϵ_1 lies between ϵ_o and ϵ_e unless the crystal has strong negative anisotropy ($\Delta\epsilon < 0$) with

$$|\Delta\epsilon| > (\epsilon_1 + \epsilon_o)^2/(3\epsilon_1 + \epsilon_o). \tag{44}$$

(These conditions would make β_j^2 negative.)

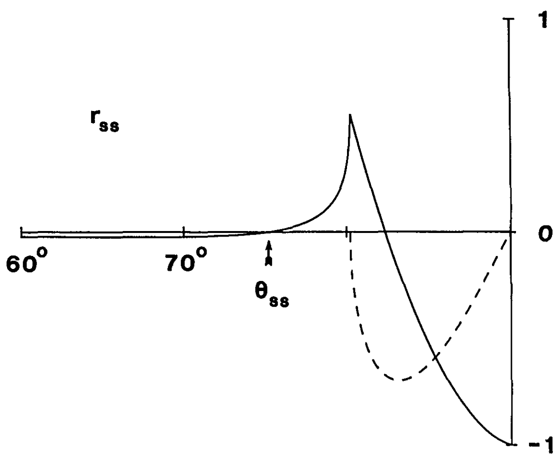

Fig. 5. Reflection amplitude r_{ss} for calcite immersed in a liquid of refractive index 1.51 and direction cosines of the optic axis given by $\alpha^2 = 0.02, \beta^2 = 0.67, \gamma^2 = 0.31$. Beyond $\theta_c^e = 80.27°$ there is an imaginary part of r_{ss} (dashed curve). The reflection amplitude is zero at $\theta_{ss} = 75.12°$.

Finally, note the surprising result that one of r_{sp} or r_{ps} will also be zero at θ_j for the given crystal orientations. This is because, from Eqs. (20), we see that when $\epsilon \to \epsilon_o$,

$$\left(\frac{cK}{\omega}\right)^2 \to \frac{\epsilon_1\epsilon_o}{\epsilon_1 + \epsilon_o}, \qquad \left(\frac{cq_o}{\omega}\right)^2 \to \frac{\epsilon_o^2}{\epsilon_1 + \epsilon_o}, \tag{45}$$

and so $q_o^2/K^2 \to \epsilon_o/\epsilon_1 = \gamma_j^2/\alpha_j^2$, so that, under the conditions of r_{pp} and r_{ss} being simultaneously zero, we also have

$$\alpha^2 q_o^2 - \gamma^2 K^2 = 0. \tag{46}$$

From Eq. (10) we see that this implies that r_{ps} will be zero if α and γ have the same sign and r_{sp} will be zero if they have opposite signs. Thus three of the four reflection amplitudes will be zero in this configuration. The square of the remaining r_{sp} or r_{ps} reflection amplitude then takes the value

$$\frac{(\epsilon_o - \epsilon_1)(\epsilon_1 - \epsilon_e)}{\epsilon_1^2 - \epsilon_1\epsilon_o + 3\epsilon_1\epsilon_e + \epsilon_o\epsilon_e}. \tag{47}$$

We see that even this can be made zero when the crystal is immersed in a medium with index n_o or n_e. For example, when $\epsilon_1 = \epsilon_e$, α_j and γ_j are zero, $\beta_j^2 = 1$ (optic axis normal to the plane of incidence), and under these conditions all reflection amplitudes are zero at angle of incidence equal to arctan (n_o/n_e).

ACKNOWLEDGMENTS

The author has enjoyed stimulating discussions of this topic with David Beaglehole and Joe Trodahl. The comments of the reviewer are also appreciated.

REFERENCES

1. M. Born and E. Wolf, *Principles of Optics*, 3rd ed. (Pergamon, Oxford, 1965), Sec. 1.5.3.
2. R. M. A. Azzam and N. M. Bashara, *Ellipsometry and Polarized Light* (North-Holland, Amsterdam, 1977), Secs. 4.2 and 4.7.3.1.
3. J. Lekner, *Theory of Reflection* (Nijhoff/Kluwer, Dordrecht, The Netherlands, 1987), Secs. 1-2 and 7-1.
4. M. Elshazly-Zaghloul and R. M. Azzam, "Brewster and pseudo-Brewster angles of uniaxial crystal surfaces and their use for determination of optical properties," J. Opt. Soc. Am. **72**, 657–661 (1982); erratum, J. Opt. Soc. Am. A **6**, 607 (1989).
5. J. Lekner, "Reflection and refraction by uniaxial crystals," J. Phys. Condens. Matter **3**, 6121–6133 (1991).
6. J. Lekner, "Bounds and zeros in reflection and refraction by uniaxial crystals," J. Phys. Condens. Matter **4**, 9459–9468 (1992).
7. M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions* (U.S. Government Printing Office, Washington, D.C., 1964), Sec. 3.8.
8. J. Lekner, "Optical properties of an isotropic layer on a uniaxial crystal substrate," J. Phys. Condens. Matter **4**, 6569–6586 (1992).