Reflection and transmission of compressional waves: Some exact results

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Exact results are derived for the reflection and transmission of acoustic compressional waves by an arbitrary stratification. These results include conservation and reciprocity theorems, low-frequency and high-frequency limiting forms, and analytic solutions for two special stratifications, both having exponential variation of density with depth, and linear or exponential variations of sound speed.

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INTRODUCTION

The purpose of this paper is to bring together exact results in the reflection and transmission of compressional waves by stratified media. In some cases we give generalizations of results recorded in acoustical texts,[1-5] in others an adaptation of techniques and results developed for electromagnetic and particle waves;[6] most of the results are new to acoustics. This paper is restricted to compressional wave propagation in planar stratifications. Although shear waves in the lower bounding medium can be incorporated into the formalism if the boundary is sharp,[7,8] we have excluded shear waves here. (Extensions of the theorems to include shear wave effects may be the subject of a future publication.) The linearized equation for the acoustic pressure p is accordingly[9,2]

$$\nabla^2 p - \frac{1}{\rho}\nabla\rho\cdot\nabla p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{1}$$

where c^2 is the adiabatic derivative of the hydrostatic pressure with respect to the density. For planar stratifications, ρ and c are functions of the depth z only; $c(z)$ is usually referred to as the local value of the phase velocity, but this is literally true only if the medium changes little in one "wavelength" ("speed"/frequency)—see Ref. 10. For a plane monochromatic wave propagating in the zx plane, solutions of (1) have the form

$$p(z,x,t) = e^{i(Kx-\omega t)}P(z), \tag{2}$$

where ω is the angular frequency of the wave and K is the x component of the wave vector, which is a constant of the motion. The angle $\theta(z)$ between the normal to the wave front and the z axis, and the local speed of sound $c(z)$ are related to K via the generalized Snell's law,

$$K = [\omega/c(z)]\sin\theta(z) = \text{const.} \tag{3}$$

(The grazing angle is $90^\circ - \theta$.) The differential equation for $P(z)$ is[8,6]

$$\rho\frac{d}{dz}\left(\frac{1}{\rho}\frac{dP}{dz}\right) + q^2P = 0, \quad q^2(z) = \frac{\omega^2}{c^2(z)} - K^2, \tag{4}$$

where $q(z)$ is the normal component of the wave vector.

In this paper we consider bounded stratifications, extending between $z = a$ and $z = b$, with uniform media (for $z < a$ and $z > b$) above and below. Acoustic parameters relating to these bounding media will be labeled by the subscripts a or b; thus $q_a = (\omega/c_a)\cos\theta_a$ is the value of the normal component of the wave vector in medium a. When sound is incident from medium a and transmitted, via the stratification, to medium b, the pressure variable $P(z)$ takes the forms

$$P_{ab} = e^{iq_az} + r_{ab}e^{-iq_az}, \quad z \leqslant a,$$
$$P_{ab} = t_{ab}e^{iq_bz}, \quad z \geqslant b, \tag{5}$$

in media a and b. When sound is incident from below, the forms taken by $P(z)$ in media a and b are

$$P_{ba} = t_{ba}e^{-iq_az}, \quad z \leqslant a,$$
$$P_{ba} = e^{-iq_bz} + r_{ba}e^{iq_bz}, \quad z \geqslant b. \tag{6}$$

In the next section we derive general relations linking the reflection amplitudes r_{ab}, r_{ba} and the transmission amplitudes t_{ab}, t_{ba}, and their relation to the appropriate acoustic reflectances and transmittances.

I. GENERAL RESULTS FOR THE REFLECTION AND TRANSMISSION AMPLITUDES

Let $F(z)$ and $G(z)$ be two solutions of (4), and consider their Wronskian

$$W(F,G) = FG' - F'G, \tag{7}$$

where the prime denotes differentiation with respect to z. The derivative of W is $W' = FG'' - F''G$, which from (4) can be written as

$$W' = (\rho'/\rho)W. \tag{8}$$

Thus W/ρ is a constant: The Wronskian of the two solutions is proportional to the local density.

Consider first the Wronskian of $P_{ab}(z)$ and $P_{ba}(z)$. From (5) and (6) we see that in medium a this takes the value $-2iq_at_{ba}$, and in medium b the value $-2iq_bt_{ab}$. Since W/ρ is a constant, we have shown that

$$Q_at_{ba} = Q_bt_{ab}, \tag{9}$$

where Q denotes the normal component of the wave vector divided by the density:

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$$Q = q/\rho. \tag{10}$$

The derivation of (9) assumes that there is no attenuation in media a and b [otherwise the forms (5) and (6) would, with complex q_a or q_b, show unacceptable exponential growth]. Total internal reflection is excluded for the same reason. It has *not* been assumed that the stratification between the uniform media a and b is free from attenuation. The equality (9) demonstrates that the (complex) transmission amplitudes t_{ab} and t_{ba} carry the same phase. It also implies that the transmittance (the fraction of acoustic intensity transmitted through the stratification) is the same from above and from below,

$$T_{ab} = T_{ba},$$

where

$$T_{ab} = (Q_b/Q_a)|t_{ab}|^2, \quad T_{ba} = (Q_a/Q_b)|t_{ba}|^2. \tag{11}$$

To see that the transmittance is $T_{ab} = (Q_b/Q_a)|t_{ab}|^2$, consider the situation in Fig. 1, in which a beam incident from medium a insonifies a strip of unit width on the $z = a$ boundary. The energy density of a plane wave with complex acoustic pressure p is proportional to $|p|^2/\rho c^2$, so the intensity is proportional to $|p|^2/\rho c$. For the case shown, the amount of energy in the primary wave that is incident on a unit area of interface in unit time is proportional to $\cos\theta_a/\rho_a c_a$, while the energy reflected away in unit time is proportional to $|r_{ab}|^2 \cos\theta_a/\rho_a c_a$. The energy transmitted in unit time is proportional to $|t_{ab}|^2 \cos\theta_b/\rho_b c_b$. It follows that

$$R_{ab} = |r_{ab}|^2,$$

$$T_{ab} = \frac{\rho_a c_a}{\rho_c c_b}\frac{\cos\theta_b}{\cos\theta_a}|t_{ab}|^2 = \frac{Q_b}{Q_a}|t_{ab}|^2. \tag{12}$$

We next consider the Wronskian of P_{ab} and P^*_{ba}, assuming the absence of attenuation everywhere [if P is a solution of (4), so is P^* provided q^2 is real]. This Wronskian takes the value $2iq_a r_{ab} t^*_{ba}$ in medium a, and $-2iq_b t_{ab} r^*_{ba}$ in medium b. Since W/ρ is constant, we have the result

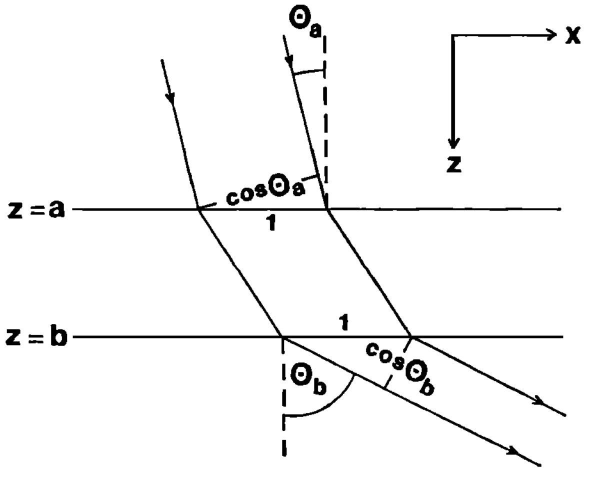

FIG. 1. A strip of unit width on the stratification is insonified by a beam of width $\cos\theta_a$; the transmitted beam has width $\cos\theta_b$ (the reflected beam is not shown).

$$Q_a r_{ab} t^*_{ba} = -Q_b t_{ab} r^*_{ba}. \tag{13}$$

With (9), this shows that

$$r^*_{ba} = -(t^*_{ba}/t_{ba}) r_{ab}, \tag{14}$$

which in turn implies the equality of the reflectances $R_{ab} = |r_{ab}|^2$ and $R_{ba} = |r_{ba}|^2$. Note that required reality of q [for Eq. (9)] also excludes total internal reflection: e.g., if the wave is totally internally reflected from medium b, q_b is imaginary.

Under the same conditions, the Wronskian of P_{ab} and P^*_{ab} is equal to $-2iq_a(1 - |r_{ab}|^2)$ in medium a, and $-2iq_b|t_{ab}|^2$ in medium b; the constancy of the Wronskian divided by the density thus implies

$$Q_a(1 - |r_{ab}|^2) = Q_b|t_{ab}|^2, \tag{15}$$

which is the energy conservation law

$$R_{ab} + T_{ab} = 1. \tag{16}$$

(The same equality links R_{ba} and T_{ba}.) The analogous results for electromagnetism and quantum mechanics are derived, by different methods, in Sec. 2-1 of Ref. 6, where references to earlier work may be found.

The differential equation (4), to be satisfied by the acoustic pressure variable $P(z)$, is linear and of the second order. Thus, in a general stratification, (4) has two linearly independent solutions, say $F(z)$ and $G(z)$, and P is a linear combination of these within the stratification:

$$P = uF + vG. \tag{17}$$

Consider the reflection-transmission problem, in which sound is incident from medium a. Then in media a and b P is given by (5), while within the inhomogeneous layer it is given by (17). The boundary conditions at $z = a$ and $z = b$ are the continuity of P and of $\rho^{-1}\,dP/dz$. (Note that these conditions are implied by the differential equation (4), and are not additional physical input.) These boundary conditions give four equations in the four unknown constants u, v, r, and t:

$$e^{i\alpha} + re^{-i\alpha} = uF_a + vG_a, \quad uF_b + vG_b = te^{i\beta},$$

$$iQ_a(e^{i\alpha} - re^{-i\alpha}) = u\overline{F}_a + v\overline{G}_a, \quad u\overline{F}_b + v\overline{G}_b = iQ_b te^{i\beta}. \tag{18}$$

Here $\alpha = q_a a$, $\beta = q_b b$, r and t are the reflection and transmission amplitudes for insonification from medium a (we will drop the ab subscript from now on), F_a is short for $F(z_a)$, $\overline{F}_a$ is short for the derivative of F at $z = a$ divided by the value of ρ just inside the stratification, and so on. (This notation allows for possible discontinuity of density at either interface). Solving (18) we find

$$r = e^{2i\alpha}[Q_a Q_b(F,G) + iQ_a(F,\overline{G}) + iQ_b(\overline{F},G) - (\overline{F},\overline{G})]/D,$$

$$t = e^{i(\alpha-\beta)} 2iQ_a(F_b\overline{G}_b - \overline{F}_b G_b)/D,$$

$$u = e^{i\alpha} 2iQ_a(\overline{G}_b - iQ_b G_b)/D,$$

$$v = -e^{i\alpha} 2iQ_a(\overline{F}_b - iQ_b F_b)/D, \tag{19}$$

where $(F,G) \equiv F_a G_b - G_a F_b$, $(F,\overline{G}) \equiv F_a\overline{G}_b - G_a\overline{F}_b$ etc., and the common denominator of all four expressions is

$$D = Q_a Q_b (F,G) + iQ_a (F,\overline{G}) - iQ_b (\overline{F},G) + (\overline{F},\overline{G}). \quad (20)$$

If the density ρ is continuous across the interfaces at $z = a$ and $z = b$, the equations linking the derivatives across the boundaries simplify, and the above equations may be replaced by a set in which $Q_{a,b} \to q_{a,b}$ and $\overline{F}, \overline{G} \to F', G'$.

In the absence of attenuation or total internal reflection, q is real everywhere. Then F and G may be chosen to be real, being the solutions of a real differential equation, and

$$R = |r|^2 = \frac{[Q_a Q_b (F,G) - (\overline{F},\overline{G})]^2 + [Q_a (F,\overline{G}) + Q_b (\overline{F},G)]^2}{[Q_a Q_b (F,G) + (\overline{F},\overline{G})]^2 + [Q_a (F,\overline{G}) - Q_b (\overline{F},G)]^2}, \quad (21)$$

$$T = \frac{Q_b}{Q_a} |t|^2 = \frac{4Q_a Q_b (F_b \overline{G}_b - \overline{F}_b G_b)^2}{[Q_a Q_b (F,G) + (\overline{F},\overline{G})]^2 + [Q_a (F,\overline{G}) - Q_b (\overline{F},G)]^2}. \quad (22)$$

By using the identity

$$(F,G)(\overline{F},\overline{G}) - (F,\overline{G})(\overline{F},G) = (F_a \overline{G}_a - \overline{F}_a G_a)(F_b \overline{G}_b - \overline{F}_b G_b) \equiv \left(\frac{W}{\rho}\right)_a \left(\frac{W}{\rho}\right)_b \quad (23)$$

and the fact that W/ρ is a constant, the conservation law (16) is seen to follow from (21) and (22). It also follows from (23) that $R \leqslant 1$, as can be seen by writing (21) in the form $1 - 4Q_a Q_b (W/\rho)^2/|D|^2$.

In total internal reflection, $q_b = (\omega/c_a) \times (c_a^2/c_b^2 - \sin^2\theta_a)^{1/2}$ is imaginary, and r takes the form $e^{2i\alpha}(iA - B)/(iA + B)$, so that $|r|^2 = 1$. But note that this is true only in the absence of attenuation, which makes the reflection less than perfect even if q_b is pure imaginary.

At grazing incidence (from medium a) the normal component $q_a = (\omega/c_a)\cos\theta_a$ of the wave vector tends to zero. It follows from (19) and (20) that

$$r \to -1, \quad t \to 0 \text{ as } \theta_a \to 90°. \quad (24)$$

[F and G are functionals of $q^2(z) = \omega^2/c^2(z) - K^2$, and thus depend on the angle of incidence through $K = (\omega/c_a)\sin\theta_a$; however, this dependence cannot override the effect of $Q_a \to 0$ in (19) and (20)]. Thus there is perfect reflection and zero transmission at grazing incidence. The reflected wave is then 180° out of phase with the incident wave. These statements hold whether or not there is attenuation in the stratification and/or the bottom medium, and also hold when there is total internal reflection (q_b imaginary).

The reflection and transmission amplitudes for insonification "from below" may be obtained by applying the boundary conditions to (6) and (17). They are

$$r_{ba} = e^{-2i\beta} [Q_a Q_b (F,G) - iQ_a (F,\overline{G}) - iQ_b (\overline{F},G) - (\overline{F},\overline{G})]/D,$$
$$t_{ba} = e^{i(\alpha - \beta)} 2iQ_b (W/\rho)_a/D, \quad (25)$$

and satisfy the reciprocity theorems (9) and (14). The corresponding solution for the constants in $P = uF + vG$ is

$$u = e^{-i\beta} 2iQ_b (\overline{G}_a + iQ_a G_a)/D,$$
$$v = -e^{-i\beta} 2iQ_b (\overline{F}_a + iQ_a F_a)/D. \quad (26)$$

The general expressions for reflection and transmission amplitudes derived here will be used in the next three sections to obtain low- and high-frequency limiting forms, and solvable models.

II. LOW-FREQUENCY REFLECTION AND TRANSMISSION

The low-frequency regime is attained when the dimensionless parameter $(\omega/c_a)\Delta z$ is small compared to unity, $\Delta z = b - a$ being the thickness of the stratification. Since this parameter is equal to $2\pi\, \Delta z/\lambda_a$, the low-frequency limit is equally well characterized as a thin-layer or long-wavelength limit. We first note that the reflection and transmission amplitudes tend to the sharp-transition values

$$r_0 = e^{2iq_a a}[(Q_a - Q_b)/(Q_a + Q_b)],$$
$$t_0 = e^{i(q_a - q_b)a}[2Q_a/(Q_a + Q_b)]. \quad (27)$$

(These are obtained by matching P and $\rho^{-1}\, dP/dz$ at $z = a$, the boundary between the uniform media a and b.) The fact that (27) gives the low-frequency limit of (19) is intuitively plausible: At long wavelengths the wave is mainly affected by the changes in the acoustical parameters, and is not sensitive to details in the transition between the two uniform media. An important question is: What are the corrections to (27) and to the reflectance and transmittance? It is natural to express the corrections as power series in Δz (more correctly, as power series in a dimensionless parameter like $\omega\Delta z/c_a$):

$$r = r_0 + r_1 + r_2 + \cdots. \quad (28)$$

A variety of techniques for extracting r_1 and the higher-order corrections are developed in Ref. 6 (see in particular Chap. 3 and Sec. 12-5). Here we will make use of the results in an accompanying paper,[11] in which expressions for r and t are given in terms of the matrix elements m_{ij} of a profile matrix M:

$$r = e^{2i\alpha} \frac{Q_a Q_b m_{12} + iQ_a m_{22} - iQ_b m_{11} + m_{21}}{Q_a Q_b m_{12} + iQ_a m_{22} + iQ_b m_{11} - m_{21}},$$
$$t = e^{i(\alpha - \beta)} \frac{2iQ_a \det M}{Q_a Q_b m_{12} + iQ_a m_{22} + iQ_b m_{11} - m_{21}} \quad (29)$$

[$\alpha = q_a a$ and $\beta = q_b b$ as in (19), to which the above expressions bear a close resemblance]. In Ref. 11 the matrix elements m_{ij} are found up to second order in the layer thickness. They are

$$m_{11} = 1 - \int_a^b dz\, \rho(z) \int_a^z d\zeta \frac{q^2(\zeta)}{\rho(\zeta)} \equiv 1 - I_2,$$
$$m_{12} = \int_a^b dz\, \rho(z) \equiv I_1,$$

$$m_{21} = -\int_a^b dz \frac{q^2(z)}{\rho(z)} \equiv -J_1,$$

$$m_{22} = 1 - \int_a^b dz \frac{q^2(z)}{\rho(z)} \int_a^z d\zeta\, \rho(\zeta) \equiv 1 - J_2. \tag{30}$$

The integrals I_1 and J_1 are of first order in $\Delta z = b - a$, while I_2 and J_2 are of second order in Δz. It is known that[11] $I_2 + J_2 = I_1 J_1$, so the determinant of the profile matrix M is unity plus a fourth-order term:

$$\det M = 1 + I_2 J_2. \tag{31}$$

The corrections to the reflectance $R = |r|^2$ depend on whether there is attenuation or not. If the attenuation is negligible within the stratification and in the uniform media, and in the absence of total internal reflection, all matrix elements and wavevectors are real, the first-order correction to $R_0 = |r_0|^2$ is zero, and the second-order correction to R_0 is

$$\begin{aligned}&[4Q_aQ_b/(Q_a+Q_b)^4][Q_a^2Q_b^2m_{12}^2 + m_{21}^2\\ &\quad + (Q_b^2 - Q_a^2)(m_{11} - m_{22}) + (Q_a^2 + Q_b^2)m_{12}m_{21}]\\ &\quad = [4Q_aQ_b/(Q_a+Q_b)^4](Q_a^2Q_b^2I_1^2\\ &\qquad + J_1^2 - 2Q_a^2J_2 - 2Q_b^2I_2).\end{aligned} \tag{32}$$

The low-frequency approximation to the reflectance, namely R_0 plus the expression (32), is shown compared to the exact reflectance in Fig. 2, for a stratification in which both density and sound speed vary exponentially with depth. For this stratification, all the integrals needed in (32) may be found analytically. The exact reflectance is calculated from the formulas of Sec. IV. Also shown is the high-frequency approximation of Sec. III.

Attenuation changes the low-frequency behavior dramatically: Whereas the correction (32) is second order in the small parameter $\omega\Delta z/c_a$, attenuation makes one of the first-order contributions to the matrix elements complex:

$$\begin{aligned}J_1 &= \int_a^b dz \frac{[k(z)^2 - K^2]}{\rho(z)}\\ &= \int_a^b dz \frac{(k_r^2 - k_i^2 - K^2 + 2ik_rk_i)}{\rho},\end{aligned} \tag{33}$$

where $\omega/c(z) \equiv k(z) = k_r(z) + ik_i(z)$. The consequence is

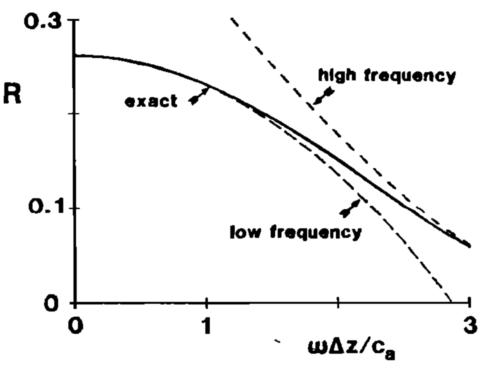

FIG. 2. Reflectance of a stratification in which density and speed vary exponentially: exact (—) from Sec. IV; low-frequency approximation $R_0 +$ (32) (– –); and high-frequency approximation of Eq. (52) (- - -). The parameters used are $\rho_b = 2\rho_a$, $c_b = (4/3)c_a$ with ρ and c continuous at $z = a$ and b, and angle of incidence 30° (60° grazing angle).

a first-order correction to the reflectance: $|r|^2$ contains the term $2\,\mathrm{Re}(r_0 r_1^*)$, and from (29) we have

$$\begin{aligned}r_0 &= e^{2i\alpha}(Q_a - Q_b)/(Q_a + Q_b),\\ r_1 &= e^{2i\alpha}[2iQ_a/(Q_a+Q_b)](J_1 - Q_b^2 I_1).\end{aligned} \tag{34}$$

The reflectivity is thus, to first order in $\omega\,\Delta z/c_a$,

$$R = R_0 - \frac{8Q_a(Q_a - Q_b)}{(Q_a+Q_b)^3}\int_a^b dz \frac{k_r(z)k_i(z)}{\rho(z)}. \tag{35}$$

Since k_r and k_i, the real and imaginary parts of $\omega/c(z)$, are both non-negative, the reflectance is decreased from R_0 if $Q_a > Q_b$, and increased from R_0 if $Q_a < Q_b$. On using $Q = \omega\cos\theta/\rho c$ and the constancy of $K = \omega\sin\theta/c$ (Snell's law), we find that $Q_a > Q_b$ if

$$\tan^2\theta_a > [(\rho_a c_a)^2 - (\rho_b c_b)^2]/[\rho_a^2(c_b^2 - c_a^2)]\,. \tag{36}$$

For stratifications in which ρ and c increase together, the right-hand side of (36) is negative, and so $Q_a > Q_b$ and attenuation in the stratification decreases the reflectance of long waves from R_0 at all angles of incidence. We note in passing that the equality $Q_a = Q_b$, which makes $R_0 = 0$, requires equality in (36), the angle at which this happens being Green's angle θ_G, the acoustical analog of Brewster's angle in optics (see Ref. 6, Sec. 1-4). At Green's angle (if it exists) the first-order correction to the reflectance vanishes. This is not true of the transmittance $T = (Q_b/Q_b)|t|^2$, which from (29), (30), and (33) is, to first order in $\omega\Delta z/c_a$,

$$T = \frac{4Q_aQ_b}{(Q_a+Q_b)^2}\left(1 - \frac{4}{Q_a+Q_b}\int_a^b dz \frac{k_r(z)k_i(z)}{\rho(z)}\right). \tag{37}$$

Thus there is a first-order correction to the transmittance at all angles, and the low-frequency attenuation correction always decreases the transmittance, as expected. [The second-order correction to T, in the absence of attenuation, is the negative of (32), since then $R + T = 1$.]

The degree of attenuation required for it to dominate the low-frequency corrections may be estimated from (32) and (35) or (37). The first- and second-order terms are, respectively, of magnitude $k_i\Delta z$ and $(\omega\Delta z/c)^2$, k_i and c here representing average values within the stratification. Attenuation is correspondingly important in the low-frequency case unless

$$k_i \ll (\omega/c)[(\omega/c)\Delta z]. \tag{38}$$

If there is attenuation in the second medium b, this will be important at all frequencies, manifesting itself in the formulas via a complex Q_b.

III. HIGH-FREQUENCY LIMITING FORMS

Reflection and transmission at high-frequency (or short-wavelength) acoustic waves is intrinsically more complicated than at the low-frequency end, because short waves are sensitive to details of the stratification while long waves are influenced by average properties as expressed in a few integrals. Nevertheless it is possible to give explicit formulas in some simple cases.

We first transform (4) by defining a new dependent variable[9] $p = (\rho_a/\rho)^{1/2}P$. The differential equation satisfied by $p(z)$ is[9,5]

$$p'' + [q^2 + \tfrac{1}{2}(\rho''/\rho) - \tfrac{3}{4}(\rho'/\rho)^2]p = 0. \qquad (39)$$

At high frequencies the q^2 term is dominant and approximate solutions of (39) are the Liouville–Green functions (see for example Ref. 6, Sec. 6-2)

$$p^{\pm}(z) = \left(\frac{q_a}{q}\right)^{1/2} e^{\pm i\varphi}, \quad \varphi(z) = \int^z d\zeta\, q(\zeta). \qquad (40)$$

The phase integral $\varphi(z)$ gives the accumulated phase at depth z; its derivative is $\varphi' = q$. [The corresponding approximations to the solutions of (4) are $P^{\pm} = (Q_a/Q)^{1/2}\exp(\pm i\varphi)$.] The Liouville–Green functions satisfy

$$(p^{\pm})'' + [q^2 + \tfrac{1}{2}(q''/q) - \tfrac{3}{4}(q'/q)^2]p^{\pm} = 0. \qquad (41)$$

In the case of acoustic waves incident from medium a, the limiting forms of $p(z)$ are

$$e^{iq_az} + re^{-iq_az} \leftarrow p(z) \rightarrow (\rho_a/\rho_b)^{1/2} t e^{iq_bz}. \qquad (42)$$

The limiting forms of p^+ are

$$e^{iq_az} \leftarrow p^+(z) \rightarrow (q_a/q_b)^{1/2} e^{i(q_bz + \varphi_+)}. \qquad (43)$$

(We have chosen the lower limit in the integral defining φ so as to make $\varphi(z) \rightarrow q_a z$ as $z \rightarrow -\infty$.) We now multiply the differential equation for p by p^+, that for p^+ by p, subtract, and integrate from $-\infty$ to $+\infty$. The result is the *comparison identity*

$$r = \frac{1}{4iq_a}\int_{-\infty}^{\infty} dz\left[\frac{q''}{q} - \frac{\rho''}{\rho} - \frac{3}{2}\left(\frac{q'}{q}\right)^2 + \frac{3}{2}\left(\frac{\rho'}{\rho}\right)^2\right]pp^+. \qquad (44)$$

This holds for all stratifications; the reflection amplitude is given as an integral over the derivatives of $q = (\omega/c_a)(c_a^2/c^2 - \sin^2\theta_a)^{1/2}$ and of ρ. The function p^+ is given by (40), while p is unknown. A useful approximation to r at high frequencies is obtained by replacing p by p^+ in (44):

$$r^{(1)} = \frac{1}{4i}\int_{-\infty}^{\infty} dz\, e^{2i\varphi}q^{-1}\left[\frac{q''}{q} - \frac{\rho''}{\rho} - \frac{3}{2}\left(\frac{q'}{q}\right)^2 + \frac{3}{2}\left(\frac{\rho'}{\rho}\right)^2\right]. \qquad (45)$$

A closely related "weak reflection" or Rayleigh approximation[12] is

$$r_R = -\int_{-\infty}^{\infty} dz\, e^{2i\varphi}\frac{Q'}{2Q}. \qquad (46)$$

This may be put in a form similar to (46) by changing to φ as integration variable, integrating by parts, and then changing back:

$$\begin{aligned} r_R &= -\frac{1}{2}\int_{-\infty}^{\infty} d\varphi\, e^{2i\varphi}Q^{-1}\frac{dQ}{d\varphi} \\ &= \frac{1}{4i}\int_{-\infty}^{\infty} d\varphi\, e^{2i\varphi}\frac{d}{d\varphi}\left(Q^{-1}\frac{dQ}{d\varphi}\right) \\ &= \frac{1}{4i}\int_{-\infty}^{\infty} dz\, e^{2i\varphi}(Q^{-1}q^{-1}Q')' \\ &= \frac{1}{4i}\int_{-\infty}^{\infty} dz\, e^{2i\varphi}q^{-1}\left[\frac{q''}{q} - \frac{\rho''}{\rho} - 2\left(\frac{q'}{q}\right)^2 + \left(\frac{\rho'}{\rho}\right)^2 + \frac{\rho'q'}{\rho q}\right]. \end{aligned} \qquad (47)$$

Both approximations fail if q is zero or small anywhere, as happens at a "classical turning point", where $c^2 = c_a^2/\sin^2\theta_a$, or at grazing incidence, when $q_a \rightarrow 0$.

As an application of (45) or (47), we will consider the high-frequency reflection amplitude from a stratification that is smooth except at a finite number of points z_j where there are discontinuities in the derivative of ρ or of c, or of both. Under these conditions the dominant terms in the integrands of (45) or (47) are delta functions at z_j, arising from the second derivatives of ρ and q. Let the density derivative ρ' change by $\Delta\rho_j'$ as z passes through z_j. This discontinuity in the derivative contributes $\Delta\rho_j'\delta(z - z_j)$ to ρ''. A discontinuity in the derivative of q gives a delta function whose strength may be calculated from

$$\frac{dq}{dz} = \frac{1}{2q}\frac{dq^2}{dz} = \frac{\omega^2}{2q}\frac{dc^{-2}}{dz} = -\frac{\omega^2}{qc^3}\frac{dc}{dz}. \qquad (48)$$

A change $\Delta c_j'$ in the derivative of c as z passes through z_j thus contributes $-(\omega^2/q_jc_j^3)\Delta c_j'\delta(z - z_j)$ to q''. The integrand near z_j thus contains the singular delta function term $-\sigma_j\delta(z - z_j)$, where the (dimensionless) strength σ_j of the delta function is determined by the discontinuities in the derivatives of density and sound speed:

$$\sigma = \Delta\rho'/q\rho + \omega^2\Delta c'/(qc)^3. \qquad (49)$$

The phase factor $\exp(2i\varphi)$ oscillates rapidly in the high-frequency limit. This ensures that smooth parts of the integrand average out to near zero, so that

$$r^{(1)} \approx \frac{i}{4}\sum_j \sigma_j e^{2i\varphi_j} \qquad (50)$$

(φ_j being the value of the phase integral at $z = z_j$). Since σ varies with frequency as ω^{-1}, the resulting reflectance is proportional to ω^{-2}, with oscillatory terms due to the phase factors $\exp(2i\varphi_j)$.

A simple and important special case is that of a stratification that is smooth except for discontinuities in the derivatives of ρ and/or c at the boundaries $z = a$ and b. The formula (50) gives

$$r^{(1)} \approx (i/4)(\sigma_a e^{2i\varphi_a} + \sigma_b e^{2i\varphi_b}), \qquad (51)$$

with the reflectance

$$R^{(1)} \approx \frac{1}{16}(\sigma_a^2 + \sigma_b^2 + 2\sigma_a\sigma_b\cos 2\Delta\varphi), \qquad (52)$$

where $\Delta\varphi$ is the increment in phase across the stratification:

$$\Delta\varphi = \varphi_b - \varphi_a = \int_a^b dz\, q(z) = \frac{\omega}{c_a}\int_a^b dz\left(\frac{c_a^2}{c^2(z)} - \sin^2\theta_a\right)^{1/2}. \qquad (53)$$

Figure 2 showed the high-frequency reflectance (52) for the exp–exp stratification discussed in Sec. IV, continuous in ρ and c at $z = a$ and $z = b$. For this case

$$\rho(z) = \rho_a e^{(z-a)/\ell}, \quad c(z) = c_a e^{(z-a)/L}, \tag{54}$$

where the lengths ℓ and L are determined in terms of the stratification thickness $\Delta z = b - a$ by

$$\ell = \Delta z/\log(\rho_b/\rho_a), \quad L = \Delta z/\log(c_b/c_a). \tag{55}$$

The strengths of the delta functions in this case are

$$\sigma_a = q_a^{-1}(\ell^{-1} + L^{-1}\sec^2\theta_a),$$
$$\sigma_b = -q_b^{-1}(\ell^{-1} + L^{-1}[1 - (c_b^2/c_a^2)\sin^2\theta_a]^{-1}). \tag{56}$$

Assuming that the angle of incidence is less than the critical angle $\theta_c = \arcsin(c_a/c_b)$, so that $q(z)$ remains real, the phase increment is

$$\Delta\varphi = L\{K[\arctan(q_b/K) - \arctan(q_a/K)] - (q_b - q_a)\}, \tag{57}$$

where $K = (\omega/c_a)\sin\theta_a$ is the tangential component of the wave vector. At normal incidence $K \to 0$, $q_a \to \omega/c_a$, $q_b \to \omega/c_b$, and (56) and (57) reduce to

$$\sigma_a = \frac{c_a}{\omega\Delta z}\log\frac{\rho_b c_b}{\rho_a c_a}, \quad \sigma_b = -\frac{c_b}{\omega\Delta z}\log\frac{\rho_b c_b}{\rho_a c_a},$$
$$\Delta\varphi = (\omega L/c_a)(1 - c_a/c_b). \tag{58}$$

Figure 3 compares the exact reflectance with the high-frequency limiting form (52), to higher values of the parameter $\omega\Delta z/c_a$ than were shown in Fig. 2. Note that the contributions from the discontinuities in the derivatives of ρ and c, which give the characteristic oscillatory decay with frequency, become dominant at quite moderate values of $\omega\Delta z/c_a$.

Although this paper is concerned with exact results and exact limiting forms, it is interesting to show in Fig. 3 the Rayleigh approximation[12] that correctly incorporates the high-frequency limiting reflectance, and also is good at low frequencies, provided the reflection is not too strong. From (46) and (54) we find, at normal incidence,

$$R_R = \tfrac{1}{4}(1 + L/\ell)^2\{[\mathrm{Ci}(\alpha) - \mathrm{Ci}(\beta)]^2 + [\mathrm{Si}(\alpha) - \mathrm{Si}(\beta)]^2\}, \tag{59}$$

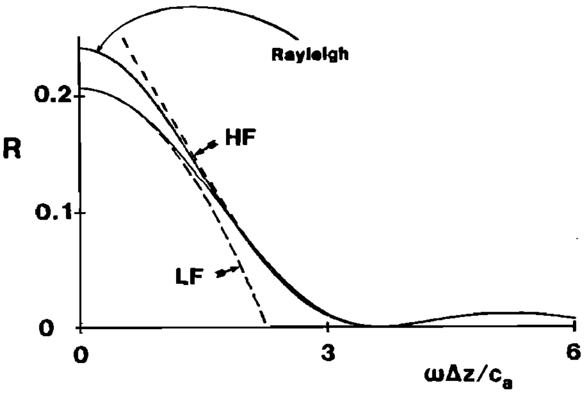

FIG. 3. Reflectance of an exp–exp stratification. The notation and parameters are as in Fig. 2, except that the results here are for normal incidence. The Rayleigh approximation is also shown (——).

where $\alpha = 2\omega L/c_a$, $\beta = 2\omega L/c_b$, and Ci and Si are the standard cosine and sine integrals.[13] The low-frequency limit of (59) is, in agreement with Eq. (34) of Ref. 12,

$$R_R \to \tfrac{1}{4}[\log(\rho_b c_b/\rho_a c_a)]^2, \tag{60}$$

which differs from the exact limit $R_0 = (\rho_b c_b - \rho_a c_a)^2/(\rho_b c_b + \rho_a c_a)^2$ by an amount of fourth order in the quantity $(\rho_b c_b - \rho_a c_a)/\rho_a c_a = x$: The leading term in the difference $R_R - R_0$ is $x^4/24$. For the case shown in Fig. 3, the ratio $\rho_b c_b/\rho_a c_a$ is not close to unity (it is 8/3), and there is a substantial difference between R_0 and the value given by (60).

IV. EXACT SOLUTIONS FOR THE EXP–LIN AND EXP–EXP STRATIFICATIONS

Several variations of the phase velocity are known for which analytic solution of the wave equation (4) is possible: Exponential decrease of sound speed with depth,[14] linear variation of speed,[10] and linear variation of the reciprocal of sound speed.[15] Here we will give solutions for exponential variation of density, with either linear or exponential variation of the sound speed. Only the latter case will be discussed in detail. For exponential density variation, with densities ρ_1 at $z = a^+$ and ρ_2 at $z = b^-$,

$$\rho(z) = \rho_1 e^{(z-a)/\ell}, \quad \ell = \Delta z/\log(\rho_2/\rho_1). \tag{61}$$

When the density is continuous at the boundaries, $\rho_1 = \rho_a$ and $\rho_2 = \rho_b$, and we regain (54) and (55). The function $p = \rho^{-1/2}P$ satisfies (39), which for exponential variation in ρ reduces to

$$p'' + [q^2 - (2\ell)^{-2}]p = 0. \tag{62}$$

If the speed of sound $c(z)$ is linear in depth, $d/dz = (\Delta c/\Delta z)(d/dc)$, and (62) may be written as

$$\frac{d^2p}{dc^2} + \left(\frac{\Delta z}{\Delta c}\right)^2\left(\frac{\omega^2}{c^2} - K^2 - \frac{1}{4\ell^2}\right)p = 0. \tag{63}$$

Comparison with Eq. 9.1.49 of Ref. 16 shows that solutions of (63) are $c^{1/2}(z)M_\nu[sc(z)]$, where M_ν is any of the modified Bessel functions $I_{\pm\nu}$ or K_ν, the order ν and "slowness" parameter s being given by

$$\nu^2 = \tfrac{1}{4} - \omega^2(\Delta z/\Delta c)^2,$$
$$s = \left|\frac{\Delta z}{\Delta c}\right|[K^2 + (2\ell)^{-2}]^{1/2}. \tag{64}$$

Since the order ν changes from real to imaginary when the angular frequency increases through $\omega_0 = \frac{1}{2}|\Delta c/\Delta z|$, two frequency ranges (0 to ω_0, and ω_0 to ∞) have to be considered separately. The formulas (19) give r and t in terms of two linearly independent solutions F and G of (4), of the form $(\rho c)^{1/2}M_\nu(sc)$.

The remainder of this section will be concerned with the "exp–exp" stratification, in which the density varies according to (61) and the sound speed according to

$$c(z) = c_1 e^{(z-a)/L}, \quad L = \Delta z/\log(c_2/c_1). \tag{65}$$

Equation (62) then reads, on transforming to the variable $x = (a - z)/L$,

$$\frac{d^2p}{dx^2} + L^2\left(\frac{\omega^2}{c_1^2}e^{2x} - K^2 - (2\ell)^{-2}\right)p = 0. \tag{66}$$

Comparison with Eq. 9.1.54 of Ref. 16 then shows that solutions of (66) are $C_\nu(\gamma e^x)$, where C_ν is any of the Bessel functions $J_{\pm\nu}$, Y_ν, and

$$\nu = L\,[K^2 + (2\ell)^{-2}]^{1/2}, \quad \gamma = L\omega/c_1. \tag{67}$$

When the order ν is not an integer, J_ν and $J_{-\nu}$ are linearly independent. (J_ν and Y_ν are always linearly independent, but the computation of Y_ν when ν is not an integer involves $J_{-\nu}$: See Ref. 16.) The simplest functions computationally are $B_{\pm\nu}$, where

$$B_\nu(y) \equiv \Gamma(\nu+1)J_\nu(y) = \left(\frac{y}{2}\right)^\nu \left(1 - \frac{y^2/4}{1.(\nu+1)} + \frac{(y^2/4)^2}{1.2(\nu+1)(\nu+2)} - \cdots\right). \tag{68}$$

The Wronskian of $B_\nu(y)$ and $B_{-\nu}(y)$ is $-2\nu/y$. In the computation of the reflection and transmission amplitudes r and t via (19), we need $B_{\pm\nu}$ and their derivatives, evaluated at $y_1 = \omega L/c_1$ and at $y_2 = \omega L/c_2$. The derivatives of $B_{\pm\nu}$ may be found from Eqs. 9.1.27 of Ref. 16; they are

$$\frac{dB_\nu}{dy} = \frac{\nu}{y}B_\nu - \frac{B_{\nu+1}}{\nu+1}, \qquad \frac{dB_{-\nu}}{dy} = \frac{\nu}{y}B_{-\nu} - \nu B_{-(\nu+1)}. \tag{69}$$

Thus the computation of $B_{\pm\nu}$ and $B_{\pm(\nu+1)}$ at two points determines the reflection and transmission amplitudes. (In the event of $\nu = n$, an integer, Y_n has to be computed via 9.1.11 of Ref. 16.)

The solutions of (4) when ν is not an integer are $F, G = \rho^{1/2}B_{\pm\nu}(\gamma e^x)$. The required functions and their derivatives are

$$F_a, G_a = B_{\pm\nu}(y_1); \quad F_b, G_b = (\rho_2/\rho_1)^{1/2}B_{\pm\nu}(y_2),$$
$$F'_a = \kappa B_\nu(y_1) + (y_1/L)\,[B_{\nu+1}(y_1)/(\nu+1)],$$
$$G'_a = \kappa B_{-\nu}(y_1) + (\nu y_1/L)B_{-(\nu+1)}(y_1),$$
$$F'_b = \left(\frac{\rho_2}{\rho_1}\right)^{1/2}\left(\kappa B_\nu(y_2) + \frac{y_2}{L}\frac{B_{\nu+1}(y_2)}{\nu+1}\right),$$
$$G'_b = (\rho_2/\rho_1)^{1/2}[\kappa B_{-\nu}(y_2) + (\nu y_2/L)B_{-(\nu+1)}(y_2)], \tag{70}$$

where $\kappa = (2\ell)^{-1} - \nu/L$.

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