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LETTERS AND COMMENTS

Quantum bouncer on a spring

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Abstract

The usual 'quantum bouncer' is a quantum particle bouncing elastically under the influence of gravity. Here we consider a particle bouncing off an impenetrable wall, and bound to it by the harmonic potential $\frac{1}{2}m\omega^2x^2$ instead of by the gravitational potential mgx. An analytic solution is possible, easily accessible to classes in intermediate quantum mechanics. The solution is related to Schrödinger's oscillating wavepacket (the original *coherent state*). Although periodic, it changes its form during the oscillation, in contrast to Schrödinger's packet. The uncertainties in the position (Δx) and the momentum (Δp) oscillate also, as does the product $\Delta x\Delta p$.

(Some figures in this article are in colour only in the electronic version)

The gravitational *quantum bouncer* [1–3] is much discussed as an elementary classical problem which in quantum mechanics has interesting features such as quantum revivals [3, 4]. The problem is to find and characterize wavepacket solutions of Schrödinger's time-dependent equation

$$\mathrm{i}\hbar\partial_t\Phi(x,t) = \left[-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right]\Phi(x,t) \tag{1}$$

for the potential $V(x) = mgx\,(x > 0)$, $V(x) = +\infty\,(x \leqslant 0)$. The solutions of Schrödinger's time-independent equation with a linear potential are Airy functions, and the energy levels E_n are proportional to the zeros of the *Ai* function [1]. The physics has led to some surprising mathematical results [2, 5]. Closed-form solutions are not available, so one is reduced to studying sums over the energy eigenstates ψ_n as solutions of (1):

$$\Phi(x,t) = \Sigma c_n\,\mathrm{e}^{-\mathrm{i}E_nt/\hbar}\psi_n(x). \tag{2}$$

In this letter we study a quantum bouncer where the force pulling the particle towards the hard wall at $x = 0$ is provided by a spring, so $V(x) = \frac{1}{2}m\omega^2x^2$ for $x > 0$. The two bouncers are illustrated in figure 1.

The superposition of any two terms in the series (2) will produce a wavefunction that oscillates at an angular frequency proportional to the energy difference between the two

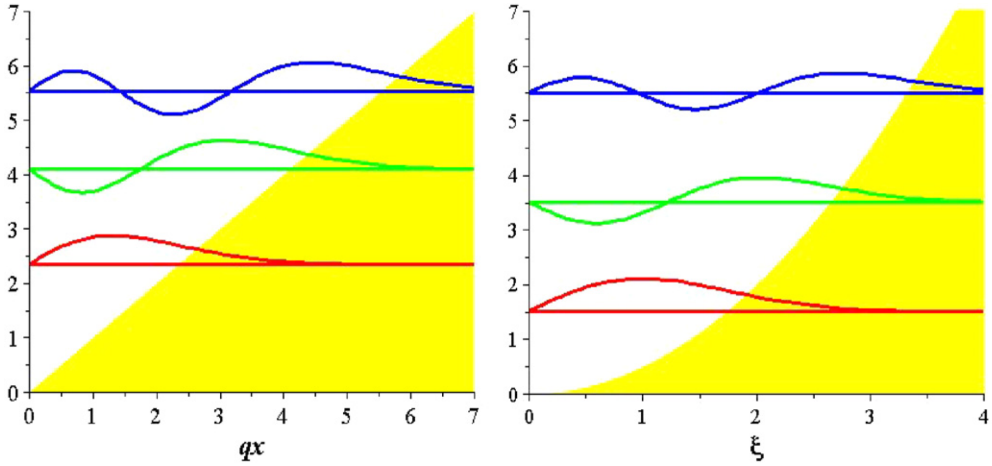

Figure 1. Quantum bouncer energy eigenstates, with gravity (on the left) and a spring (on the right) providing the restoring force. In both cases there is an impenetrable wall at $x = 0$, $V(x) = +\infty$ for $x \leqslant 0$. The first three energy levels are shown. The energy eigenstates are Airy functions (on the left), and the odd harmonic oscillator eigenstates (on the right). The energy scales are $\frac{\hbar^2 q^2}{2m}$ on the left, where $q = (\frac{2m^2 g}{\hbar^2})^{\frac{1}{3}}$ and g is the acceleration due to gravity, and $\hbar\omega$ is on the right. The horizontal scale on the right is $\xi = (m\omega/\hbar)^{1/2}x$.

eigenstates being superposed. This is not the oscillating wavepacket we are interested in: we wish to construct a wavepacket which oscillates at the classical frequency (which depends on the displacement amplitude in the gravitational case). For this we need a smooth superposition of infinitely many eigenstates. This is precisely what Schrödinger constructed for the full harmonic potential in the early days of quantum mechanics [6, 7]. His wavepacket is

$$\Phi_S(x,t) = \exp\left\{-\frac{m\omega}{2\hbar}\left[(x - x_m\cos\omega t)^2 + 2\mathrm{i}xx_m\sin\omega t - \frac{\mathrm{i}}{2}x_m^2\sin 2\omega t\right] - \frac{\mathrm{i}}{2}\omega t\right\}. \tag{3}$$

(Schrödinger actually gave an expression equivalent to the real part of (3); he used time factors $\mathrm{e}^{+\mathrm{i}E_n t/\hbar}$ rather than $\mathrm{e}^{-\mathrm{i}E_n t/\hbar}$, the time-dependent Schrödinger equation not having yet been formulated.) The absolute square of (3) is

$$|\Phi_S(x,t)|^2 = \exp\left\{-\frac{m\omega}{\hbar}(x - x_m\cos\omega t)^2\right\} \tag{4}$$

and thus Φ_S represents oscillatory motion within the harmonic potential well, without change in its envelope, and with the classical period $T = 2\pi/\omega$. Note however that the full complex Φ_S repeats after *twice* the classical period. This is analogous to the precession of spin-1/2 in a magnetic field, in which the expectation value of the spin rotates about the field direction at the Larmor frequency, while the spinor itself takes two Larmor cycles to return to its original value.

A formal solution to the general bouncer problem, where a wavepacket solution $\Phi(x,t)$ is known for any even potential $V(x)$ *without the wall at origin*, can be written down easily [8] as

$$\Psi(x,t) = \Phi(x,t) - \Phi(-x,t). \tag{5}$$

(Since $V(-x) = V(x)$, $\Phi(-x,t)$ is a solution of (1) if $\Phi(x,t)$ is a solution, and (5) is zero at $x = 0$, thus satisfying the impenetrable wall boundary condition.) A similar idea (in conjunction with supersymmetric quantum theory) was used to construct a wavepacket totally reflected by the potential $\hbar^2/mx^2$ [9].

For the spring-held quantum bouncer, we can take $\Phi = \Phi_S$, or indeed any wavepacket solution of the time-dependent Schrödinger equation for the full harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$, $-\infty < x < \infty$. Since $\Psi(x, t)$ is odd in x, it can be expressed as a sum over the odd energy eigenstates of the full harmonic oscillator. In the remainder of this letter we shall explore the properties of the bouncing wavepacket $\Psi_S(x,t) = \Phi_S(x,t) - \Phi_S(-x,t)$.

We first note some of the properties of Schrödinger's wavepacket Φ_S. The expectation values of energy, position and momentum have (apart from the $\frac{1}{2}\hbar\omega$ term in the energy) exactly the classical form:

$$\begin{aligned} \langle E\rangle_S &= \tfrac{1}{2}\hbar\omega + \tfrac{1}{2}m\omega^2x_m^2 \\ \langle x\rangle_S &= x_m\cos\omega t \\ \langle p\rangle_S &= -m\omega x_m\sin\omega t. \end{aligned} \tag{6}$$

The variances in the position and momentum are both constant in time:

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}, \qquad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{1}{2}m\hbar\omega. \tag{7}$$

The uncertainty product takes its minimum possible value: $\Delta x\Delta p = \hbar/2$. It is well known that Schrödinger's wavepacket is a special case of the coherent states of quantum optics [10, 11].

When the hard wall is inserted, all of the above expectation values are changed, because the wavepacket must now change shape as it oscillates. Figure 2 shows two snapshots of the wavepacket Ψ_S.

The bouncing wavepacket has interference zeros at odd multiples of $T/4$, with $T = 2\pi/\omega$ being the classical period of the full harmonic oscillator and $T/2$ being the classical period of the bouncer. (Since the complex Φ_S returns to its original value after time $2T$, there are four bounces before the complex Ψ_S repeats.) The interference zeros are located at multiples of $\pi\hbar/m\omega x_m$; one example is shown in the lower diagram of figure 2.

Let $\xi = (m\omega/\hbar)^{1/2}x$ be the dimensionless measure of displacement, and $\xi_m = (m\omega/\hbar)^{1/2}x_m$ correspond to the maximum displacement. Then the expectation values of energy, position and momentum are

$$\begin{aligned} \langle E\rangle &= \frac{1}{2}\hbar\omega\frac{\left(1+\xi_m^2\right)e^{\xi_m^2} + \xi_m^2 - 1}{e^{\xi_m^2} - 1} \\ \langle x\rangle &= x_m\frac{\cos\omega t\,\mathrm{erf}(\xi_m\cos\omega t)\,e^{\xi_m^2} - i\sin\omega t\,\mathrm{erf}(i\xi_m\sin\omega t)}{e^{\xi_m^2} - 1} \\ \langle p\rangle &= -m\omega x_m\frac{\sin\omega t\,\mathrm{erf}(\xi_m\cos\omega t)\,e^{\xi_m^2} + i\cos\omega t\,\mathrm{erf}(i\xi_m\sin\omega t)}{e^{\xi_m^2} - 1}. \end{aligned} \tag{8}$$

The energy expectation value tends for large ξ_m to that given in (6) for Φ_S, namely $\frac{1}{2}\hbar\omega\left(1+\xi_m^2\right)$, and to $\frac{3}{2}\hbar\omega$ for small ξ_m. In accord with Ehrenfest's theorem (see for example [7], section 7), $\langle p\rangle$ is equal to m times the time derivative of $\langle x\rangle$. Both $\langle x\rangle$and $\langle p\rangle$ oscillate with the period $T/2$, $\langle x\rangle$ about a mean which increases with x_m to its asymptotic value $(2/\pi)\,x_m$, $\langle p\rangle$ about zero. Figure 3 shows the position and momentum expectation values for a moderately large value of the dimensionless displacement, so chosen to illustrate the approach to classical motion.

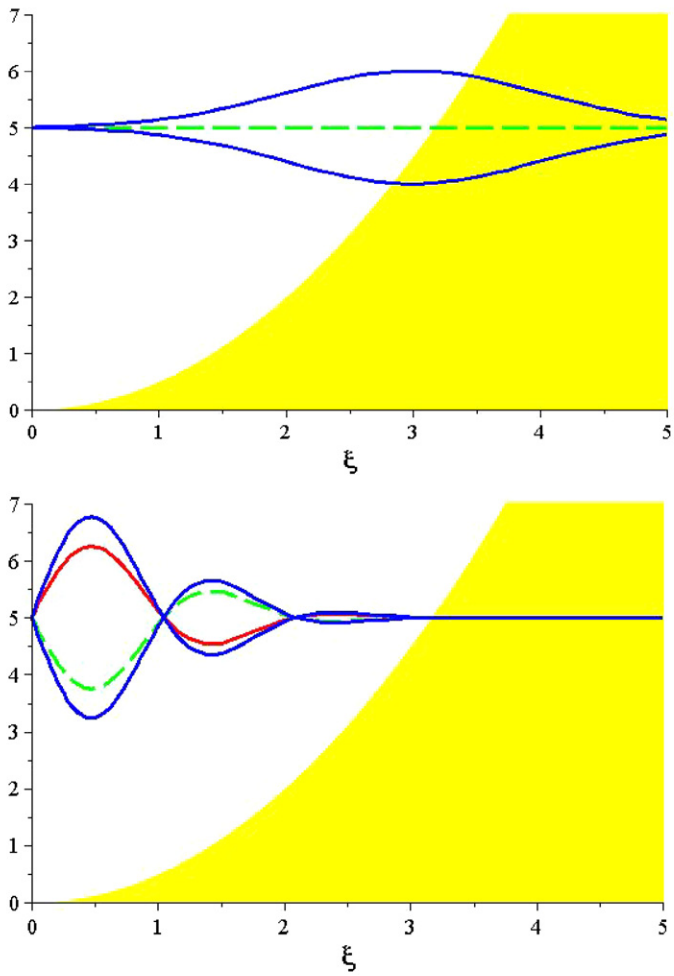

Figure 2. The quantum bouncer with the dimensionless displacement amplitude $\xi_m = (m\omega/\hbar)^{1/2} x_m$ equal to 3, at $t = 0$ (top) and $t = 3T/4$ (bottom). The envelope (blue), real part (red) and imaginary part (green, dashed) of Ψ_S are shown. The energy scale is in units of $\hbar\omega$. Animations of this wavepacket and of the Schrödinger wavepacket can be viewed at http://www.victoria.ac.nz/scps/staff/johnlekner/animations.aspx.

The expectation values of the squares of the position and momentum do not contain the error function:

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \frac{\left[2\xi_m^2 \cos^2 \omega t + 1\right] e^{\xi_m^2} + 2\xi_m^2 \sin^2 \omega t - 1}{e^{\xi_m^2} - 1}$$
$$\langle p^2 \rangle = \frac{m\hbar\omega}{2} \frac{\left[2\xi_m^2 \sin^2 \omega t + 1\right] e^{\xi_m^2} + 2\xi_m^2 \cos^2 \omega t - 1}{e^{\xi_m^2} - 1}. \quad (9)$$

Both oscillate with the period $T/2$, the classical period of the bouncer. The uncertainty product $\Delta x \Delta p$ is now time dependent, and oscillates between small and large values, again

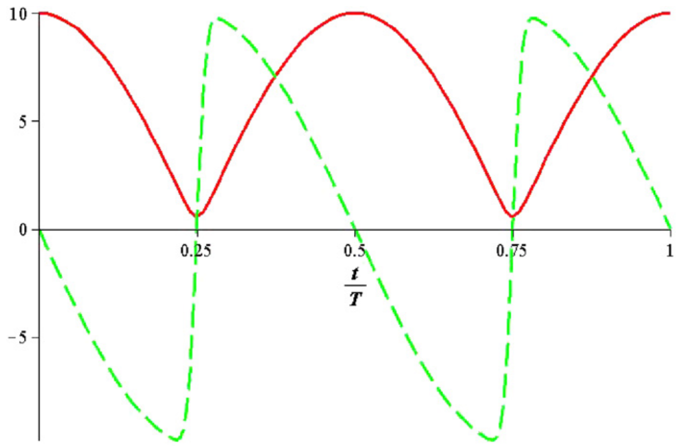

Figure 3. Time variation of $\langle x\rangle\,(m\omega/\hbar)^{1/2}$ (red, solid curve) and $\langle p\rangle/(m\omega\hbar)^{1/2}$ (green, dashed curve), drawn for $\xi_m = (m\omega/\hbar)^{1/2}x_m = 10$. At this value of ξ_m the expectation values are already close to the classical displacement and momentum, which are $x(t) = x_m|\cos\omega t|$ and m times the time derivative of $x(t)$.

at the classical bouncer frequency. When the dimensionless maximum displacement is large compared to unity, the small and large values take simple forms:

$$\begin{aligned}\Delta x\,\Delta p &\to \frac{\hbar}{2} \qquad \text{at} \quad t = 0, \quad \pi/\omega, \quad 2\pi/\omega, \ldots \\ \Delta x\,\Delta p &\to \frac{\hbar}{2}\xi_m\left(\frac{2\pi-4}{\pi}\right)^{1/2} \qquad \text{at} \quad t = \pi/2\omega, \quad 3\pi/2\omega, \ldots.\end{aligned} \tag{10}$$

The small value, which is the minimum allowed by quantum mechanics, occurs at the maximum displacement. The large value is associated with the interference fringes between what we can regard as the incoming and reflected wavepackets, $\Phi_S(\pm x, t)$. The mean square deviations $(\Delta x)^2$ and $(\Delta p)^2$ are out of phase, with the maximum Δx coincident with the minimum in Δp, as expected. The time variation of $\Delta x\Delta p$ is more complicated, as shown in figure 4.

Finally, let us consider the autocorrelation function $A(t)$, where [4]

$$A(t) = \int_0^\infty \mathrm{d}x\,\Psi^*(x,0)\Psi(x,t) \tag{11}$$

(a normalized Ψ is assumed). If we substitute an expansion in terms of the (normalized) energy eigenstates for Ψ, as in (2), we get

$$A(t) = \Sigma|c_n|^2\,\mathrm{e}^{-\mathrm{i}E_n t/\hbar}. \tag{12}$$

When a packet exactly matches its initial value, $|A(t)|^2 = 1$. For the Schrödinger wavepacket (3) of the full harmonic oscillator,

$$|A_S(t)|^2 = \exp\left\{-\frac{m\omega}{\hbar}x_m^2(1-\cos\omega t)\right\}. \tag{13}$$

Thus, even though this packet is coherent and unchanging in its envelope, the absolute square of the autocorrelation function varies from unity (at $t = 0$ or integer multiples of the classical period T) to $\mathrm{e}^{-\frac{2m\omega}{\hbar}x_m^2} = \mathrm{e}^{-2\xi_m^2}$ at odd multiples of $T/2$. For large ξ_m this is very small, indicating very little overlap between $\Phi_S(x,0)$ and $\Phi_S(x,T/2)$.

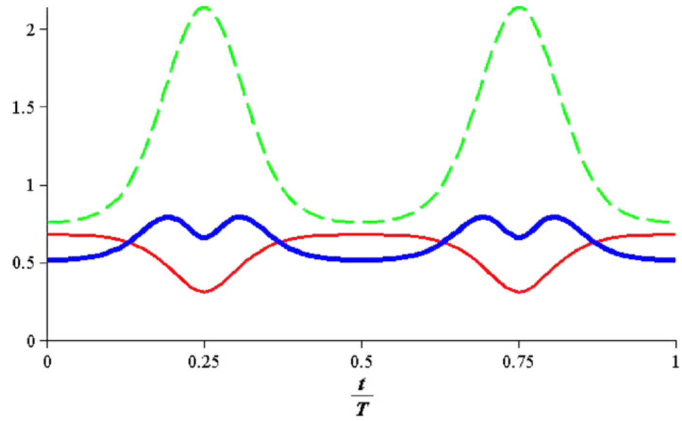

Figure 4. Time variation of the uncertainties $\Delta p/(m\omega\hbar)^{1/2}$ (green, dashed top curve), $\Delta x(m\omega/\hbar)^{1/2}$ (red) and $\Delta x\Delta p/\hbar$ (blue thick curve). Note that Δp is maximum at odd multiples of T/4, when the interference zeros occur. The product $\Delta x\Delta p$ is approximately $\hbar/2$ at integer multiples of T/2, and is maximum on either side of odd multiples of T/4. The curves are drawn for $\xi_m = 2$. For large ξ_m the maxima in $\Delta x\Delta p$ form a narrow plateau centred on odd multiples of T/4.

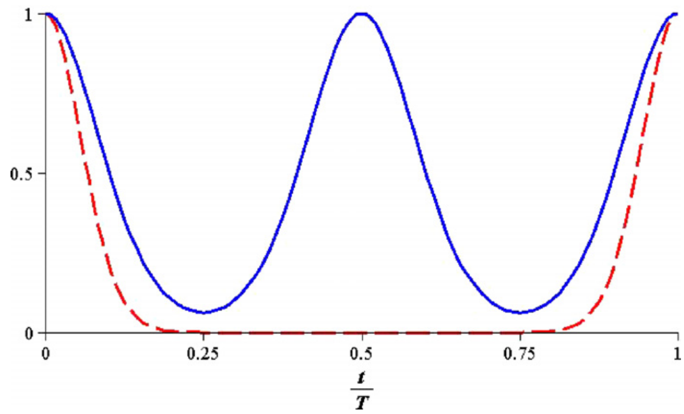

Figure 5. The absolute square of the autocorrelation function, $|A(t)|^2$, for the Schrödinger wavepacket Φ_S (red dashed curve) and the quantum bouncer Ψ_S (blue solid curve). Both curves are drawn for $\xi_m = 2$.

We expect $|A(t)|^2$ to vary strongly during a cycle of the harmonic bouncer, and it does:

$$|A(t)|^2 = \frac{2\,\mathrm{e}^{\xi_m^2}}{\left(\mathrm{e}^{\xi_m^2} - 1\right)^2}\left\{\cosh\left(\xi_m^2 \cos\omega t\right) - \cos\left(\xi_m^2 \sin\omega t\right)\right\}. \tag{14}$$

The autocorrelation is unity at $t = 0$ and at multiples of $T/2 = \pi/\omega$, the classical period of the bouncer. In between, at odd multiples of T/4, it becomes exponentially small when ξ_m is large compared to unity:

$$\left|A\left(\frac{\pi}{2\omega}\right)\right|^2 = \left[\frac{\sin\left(\xi_m^2/2\right)}{\sinh\left(\xi_m^2/2\right)}\right]^2. \tag{15}$$

Figure 5 illustrates the time variation of the autocorrelation function for the full harmonic oscillator and for the bouncer on a spring.

To sum up: we have given an explicit solution to the harmonic quantum bouncer, constructed from Schrödinger's original coherent wavepacket. It is perfectly periodic, but with a strong variation in its uncertainty product. It is too simple to show the more subtle phenomenon of quantum revival, but interesting nevertheless in its correlations and time-varying waveform.

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