Nonreflecting stratifications

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Received October 7, 1989

Electromagnetic, acoustic, and particle waves are specularly reflected by planar inhomogeneities (stratifications). A brief review is given of the cases in which zero reflection is known to occur. In general these are rare, but stratifications that are symmetric with respect to inversion about their midplane have an analytic property of the reflection amplitude that makes zero reflection a frequent occurrence. Examples are given of the locii of zero reflection in the angle of incidence – thickness plane of some symmetric stratifications. For uniform layers the zero reflection locii can be classified into Malus–Brewster–Green type and interference type, but this distinction does not hold in general.

Les ondes électromagnétiques, les ondes acoustiques et les ondes de particules sont réfléchies spéculairement par des inhomogénéités planaires (stratifications). On présente une brève revue des cas où l'on sait que la réflexion est nulle. En général, ces cas sont rares, mais les stratifications qui sont symétriques par rapport à l'inversion dans leur plan médian ont une propriété analytique de l'amplitude de réflexion qui fait que la réflexion nulle est fréquente. On donne des exemples des lieux de réflexion nulle dans le plan angle d'incidence-épaisseur de certaines stratifications symétriques. Pour des couches uniformes, les lieux de réflexion nulle peuvent être classés en deux types: type Malus–Brewster–Green et type interférence. Cette distinction ne tient toutefois pas en général.

[Traduit par la revue]

Can. J. Phys. 68, 738 (1990)

1. Introduction

Consider two uniform media, and a planar transition region between them. If the speed of propagation of harmonic waves, and other relevant properties (such as the density in the case of sound waves) are functions of z only, where z is the distance measured normal to the interface, separation of variables reduces the partial differential equation describing the wave motion to an ordinary differential equation. The separation of variables constant is the square of the wave vector component K parallel to the interface (along the x direction, say), and the constancy of K implies Snell's law

$$[1] \qquad K = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

where subscripts 1 and 2 denote the uniform media on either side of the stratification, θ_1 is the angle of incidence, θ_2 the angle of refraction, and k is the magnitude of the total wave vector. The geometry is shown in Fig. 1.

If ω is the angular frequency of the harmonic wave, the total wave function for a plane wave propagating in the z-x plane is

$$[2] \qquad \Psi(z,x,t) = \psi(z) \exp i(Kx - \omega t)$$

The function $\psi(z)$ has the limiting forms

$$[3] \qquad e^{iq_1z} + r\, e^{-iq_1z} \longleftarrow \psi \longrightarrow t\, e^{iq_2z}$$

where $q_1 = k_1 \cos\theta_1$ and $q_2 = k_2 \cos\theta_2$ are the normal components of the wave vector in media 1 and 2. Relation [3] defines the reflection and transmission amplitudes r and t. The differential equation satisfied by ψ depends on the type of wave under consideration, but for particle, electromagnetic, and acoustic compressional waves it may be put in the form

$$[4] \qquad \frac{d^2\psi}{dz^2} - \frac{1}{f}\frac{df}{dz}\frac{d\psi}{dz} + q^2\psi = 0, \qquad q^2(z) = k^2(z) - K^2$$

where ψ, f, and k^2 are given in Table 1.

The above is a unification of results that may be found in various texts and papers. We give only one reference each for quantum mechanics (1), optics (2) and acoustics (3); a unified treatment of reflection phenomena has been given in a recent monograph (4), in which many more references are listed.

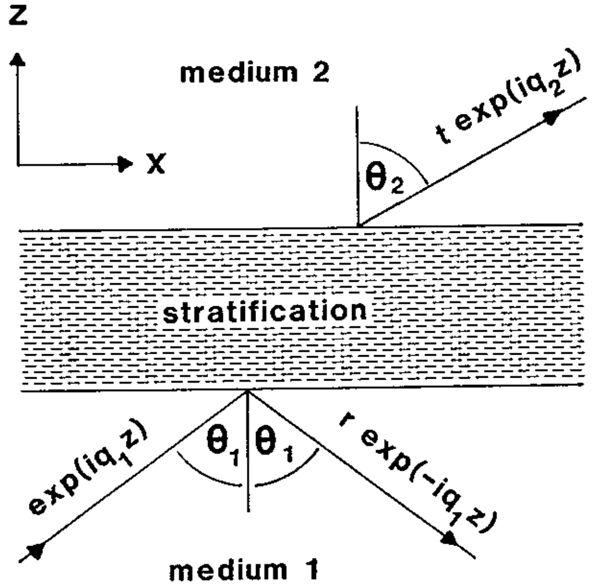

Fig. 1. Reflection and transmission by a planar stratification. The directions and wave forms of the incident, reflected, and transmitted waves are shown, with the common factor $\exp[i(Kx-\omega t)]$ suppressed.

A question of practical and theoretical interest is: under what conditions does an interface not reflect at all? General considerations suggest that such a situation should be rare, since the reflection amplitude r is a complex number (its magnitude squared gives the reflected intensity, and its phase is important in phenomena involving the superposition of waves, such as in the reflection of beams or pulses; see Chap. 10 of ref. 4). As the parameters of the reflection problem, such as the angle of incidence, the thickness of the stratification, or the stratification "profile" are varied, the reflection amplitude r will move within the unit circle in the complex plane. A zero of r requires both the real and imaginary parts to be zero simultane-

TABLE 1. Characterization of variables in eq. [4] for four types of wave. Of the symbols not defined in the text, m is the particle mass, E and $V(z)$ the particle total and potential energies, c is the speed of light, and $v(z)$ is the local speed of sound

Wave	ψ	f	k^2
Particle	Probability amplitude	Constant	$\frac{2m(E-V)}{\hbar^2}$
Electromagnetic, s polarization	Electric field	μ (magnetic permeability)	$\varepsilon\mu\,\omega^2/c^2$
Electromagnetic, p polarization	Magnetic field	ε (dielectric function)	$\varepsilon\mu\,\omega^2/c^2$
Acoustic	Acoustic pressure	ρ (density)	ω^2/v^2

ously, which is seen to require a special set of circumstances, or an unlikely coincidence. In the next section we consider some examples of profiles that are known to be nonreflecting somewhere in their parameter space, and then we will discuss a special class, namely symmetric profiles, which have been shown to be likely candidates for nonreflecting stratifications.

2. Examples of zero r

In the year 1810, the celebrated French philosopher M. Malus, while looking through a prism of calcareous spar at the light of the setting sun reflected from the windows of the Luxembourg palace in Paris, was led to the curious discovery, that a beam of light reflected from *glass* at an angle of 56°, or from *water* at an angle of 52°45′, possessed the very same properties as one of the rays formed by a rhomb of calcareous spar; that is, that it was wholly polarised ...

This description of the discovery of polarization by reflection is quoted from Brewster's *Treatise on optics* (5); the italics are Brewster's. While appreciating the romance of the setting, one should note that Malus published his discovery in 1809, and that he gives the polarizing angle for glass at 56° 30′ (6, 7). Brewster found, as the result of extensive experiments, that "the index of refraction is the tangent of the angle of polarisation" (see ref. 5 chap. 24), and this angle is now known as Brewster's angle.

The importance of this bit of history to the subject at hand is that polarization by reflection occurs because one of the components of light, the p polarization, is not reflected at the polarizing angle. Thus polarization by reflection is the first known instance of *zero reflection* from the specular surface. A related case of zero reflection was soon to follow for acoustic compressional waves, although in this case polarization is not involved (see ref. 8 and Sect. 1.4 of ref. 4). These two cases of zero reflection can be considered together, by noting that [4] can be written as

$$[5] \qquad f\frac{d}{dz}\left(\frac{1}{f}\frac{d\psi}{dz}\right)+q^2\psi=0$$

which shows that $f^{-1}(d\psi/dz)$ as well as ψ must be continuous at a discontinuity in k^2 and (or) f, such as occurs at the boundary of two media. Applying the continuity of ψ and ψ'/f at a boundary (sharp interface) between media 1 and 2, located for convenience at $z=0$, and using [3], we find

$$[6] \qquad 1+r_0=t_0, \qquad Q_1(1-r_0)=Q_2t_0$$

where $Q_1 = q_1/f_1$, $Q_2 = q_2/f_2$, and r_0 and t_0 denote the reflection and transmission amplitudes for this case (r_0 and t_0 are known as the Fresnel coefficients in optics). From [6],

$$[7] \qquad r_0=\frac{Q_1-Q_2}{Q_1+Q_2}, \qquad t_0=\frac{2Q_1}{Q_1+Q_2}$$

Thus, for a sharp interface between media 1 and 2, zero reflection occurs when

$$[8] \qquad Q_1=Q_2 \quad \text{or} \quad f_1\tan\theta_1=f_2\tan\theta_2$$

Equation [8] together with Snell's law [1] and $f=\varepsilon$ for the electromagnetic p wave give Brewster's relation for the angle θ_1 at which r_0 is zero

$$[9] \qquad \tan^2\theta_B=\frac{\varepsilon_2}{\varepsilon_1}$$

(we have assumed the magnetic permeability μ to be the same for both media, in which case the refractive index is $(\varepsilon_2/\varepsilon_1)^{1/2}$). The same relations with $f=\rho$ for the acoustic case give Green's angle

$$[10] \qquad \tan^2\theta_G=\frac{(\rho_2v_2/\rho_1v_1)^2-1}{1-(v_2/v_1)^2}$$

The Malus–Brewster and Green zero reflectances occur because the wave experiences no change in Q, the effective normal component of the wave vector (see Sections. 1.2 and 1.4 of ref. 4).

Until recently, all other known zero reflectances were of the destructive interference type, of which the best known is the antireflection coating: a uniform layer interposed between the media 1 and 2, with thickness and refractive index chosen so that the reflection from the front face is matched by an equal in magnitude but opposite in sign reflection originating at the back face (see Sect. 2.4 of ref. 4). Other stratifications that are known to give destructive interference reflection zeros are the Raleigh profile, the sech^2 profile, dielectric layers on absorbing substrates, and the fascinating phenomenon of attenuated total reflection. These are discussed in Sects. 2.5, 4.3, 8.3, and 8.6 of ref. 4. In the next section we discuss symmetric stratifications, which were recently shown to have a strong likelihood of exhibiting zero reflection (9). In Sect. 4 we show that in general there is no clear division between the Malus–Brewster–Green type of zero and the destructive interference type of zero.

3. Symmetric stratifications

The differential equation [4] can be written as

$$[11] \qquad L\psi=0, \qquad L=\frac{d^2}{dz^2}-\frac{1}{f}\frac{df}{dz}\frac{d}{dz}+q^2$$

L is even in z if f and q^2 are even in z; then if $\psi(z)$ is a solution, so is $\psi(-z)$ and (since the equation is linear) so are $\psi(z) \pm \psi(-z)$. Thus, if f and q^2 are even in z, the two linearly independent solutions of [4] can be taken to be even and odd, respectively. (In the language of quantum mechanics, parity is a good quantum number.) We shall see shortly that there are important consequences of inversion symmetry for reflection. But first we will derive a general expression for the reflection amplitude r from a stratification that extends from z_1 to z_2. For a plane wave incident from medium 1 (occupying the half-space $z < z_1$), the wave function $\psi(z)$ takes the form

$$\psi(z) = \begin{cases} e^{iq_1 z} + r\, e^{-iq_1 z}, & z < z_1 \\ \alpha F(z) + \beta G(z), & z_1 \le z \le z_2 \\ t\, e^{iq_2 z}, & z > z_2 \end{cases} \tag{12}$$

where F and G are two linearly independent solutions of [4] in the stratification, and the four constants r, α, β, and t are to be determined by satisfying the boundary conditions at z_1 and z_2. These are that ψ and $f^{-1}\, d\psi/dz$ be continuous (as noted below [5]). We thus have four linear equations in the four unknowns; solving for r we find

$$r = e^{2iq_1 z_1} \times \frac{Q_1 Q_2 (F,G) + iQ_1 (F,\bar{G}) + iQ_2(\bar{F},G) - (\bar{F},\bar{G})}{Q_1 Q_2 (F,G) + iQ_1 (F,\bar{G}) - iQ_2(\bar{F},G) + (\bar{F},\bar{G})} \tag{13}$$

where $Q_1 = q_1/f_1$, $Q_2 = q_2/f_2$ as before,

$$\begin{aligned} (F,G) &\equiv F_1 G_2 - G_1 F_2, \\ (F,\bar{G}) &\equiv F_1 \bar{G}_2 - G_1 \bar{F}_2, \qquad \text{etc} \end{aligned} \tag{14}$$

with $F_1 = F(z_1)$ etc, $\bar{F}_1$ denoting the derivative of F at z_1 divided by the value of f just inside the stratification. (This notation allows for the discontinuity of f at the boundaries. If f is continuous at z_1 and z_2, the numerator and denominator of [13] both have the common factor $(f_1 f_2)^{-1}$, and in [13] one can replace Q_1, Q_2 by q_1, q_2 and $\bar{F}$, $\bar{G}$ by the derivatives of F and G.) For a uniform layer, F and G can be taken to be $\exp(\pm iqz)$ or $\cos qz$ and $\sin qz$, and the well-known formulae of acoustics and optics are regained (see Sect. 3 of ref. 3 and Sect. 2.4 of ref. 4).

We now return to symmetric stratifications, for which we choose the origin of z at the plane of inversion symmetry. Then, as noted at the beginning of this section, the solutions F and G can be chosen to be even and odd, respectively: $F(-z) = F(z)$, $G(-z) = -G(z)$, $F'(-z) = -F'(z)$, and $G'(-z) = G'(z)$ (primes denote differentiation with respect to z). Thus, for symmetric stratifications,

$$\begin{aligned} (F,G) &= 2F_2G_2, \qquad (\bar{F},\bar{G}) = -2\bar{F}_2\bar{G}_2 \\ (F,\bar{G}) &= F_2\bar{G}_2 + G_2\bar{F}_2 = -(\bar{F},G) \end{aligned} \tag{15}$$

From now on we restrict ourselves to full inversion symmetry for the reflection problem, so that media 1 and 2 are identical. We denote by q_0 and Q_0 the common values of q_1, q_2 and of Q_1, Q_2, etc. Then, using [15], the expression [13] for the reflection amplitude reduces to

$$r = e^{2iq_0 z_1} \frac{Q_0^2 F_2 G_2 + \bar{F}_2 \bar{G}_2}{Q_0^2 F_2 G_2 + iQ_2(F_2\bar{G}_2 + G_2\bar{F}_2) - \bar{F}_2\bar{G}_2} \tag{16}$$

In contrast to [13], the numerator of this expression is real. Thus, for symmetric stratifications it is only necessary for one real quantity, namely $Q_0^2 F_2 G_2 + \bar{F}_2\bar{G}_2$, to be zero for r to be zero, whereas in the general case both the real and imaginary parts of the numerator have to be zero simultaneously for null reflectance.

In the next section we will give some examples of symmetric stratifications and their zero-reflectance locii. We conclude this section with a useful approximation to the functions F and G, namely, the Liouville–Green functions

$$\chi_\pm(z) = q(z)^{-1/2}\, e^{\pm i\phi(z)}, \qquad \phi(z) = \int^z d\zeta\, q(\zeta) \tag{17}$$

The properties of the Liouville–Green approximate wave forms are discussed in more detail in Sect. 6.2 of ref. 4, where also a brief historical explanation is given for the name used (as opposed to the more common WKB or JWKB). Here we note only that the differential equation satisfied by $\chi_\pm$ is

$$\chi''_\pm + \left[q^2 + \frac{1}{2}\frac{q''}{q} - \frac{3}{4}\left(\frac{q'}{q}\right)^2 \right] \chi_\pm = 0 \tag{18}$$

while the substitution $\psi = f^{1/2}\chi$ brings [4] to the form

$$\chi'' + \left[q^2 + \frac{1}{2}\frac{f''}{f} - \frac{3}{4}\left(\frac{f'}{f}\right)^2 \right] \chi = 0 \tag{19}$$

The terms accompanying q^2 in the square brackets of [18] and [19] are of order l^{-2}, where l is a length characterizing the variation of physical parameters within the stratification. Provided the dimensionless quantity $(ql)^2$ is large, both [18] and [19] are approximately of the form $\chi'' + q^2\chi = 0$, and thus the Liouville–Green functions are good approximations to $\chi = f^{-1/2}\psi$, so that solutions of [4] are well represented by

$$\psi_\pm = (f(z)/q(z))^{1/2}\, e^{\pm i\phi(z)} \equiv Q(z)^{-1/2}\, e^{\pm i\phi(z)} \tag{20}$$

(As discussed in Chap. 6 of ref. 4, these approximations fail when q is small, as happens near grazing incidence where $q_1 \to 0$, or near a classical turning point where $q(z) \to 0$. Special techniques are then needed.)

For the symmetric stratifications under consideration, f and q^2 are even under inversion. If the lower limit of integration in the definition of the phase integral ϕ in [17] is taken to be in the plane of symmetry $z = 0$, ϕ is also even under inversion. Then the appropriate approximations to the even and odd functions F and G are

$$F \approx Q^{-1/2}\cos\phi, \qquad G \approx Q^{-1/2}\sin\phi \tag{21}$$

From [21] it follows that

$$\begin{aligned} \bar{F} &\equiv F'/f = -Q^{1/2}(\sin\phi + \tfrac{1}{2}\Gamma\cos\phi) \\ \bar{G} &\equiv G'/f = Q^{1/2}(\cos\phi - \tfrac{1}{2}\Gamma\sin\phi) \end{aligned} \tag{22}$$

where the dimensionless function Γ is q^{-1} times the logarithmic derivative of Q:

$$\Gamma \equiv \frac{Q'}{fQ^2} = \frac{Q'}{qQ} = \frac{q'}{q^2} - \frac{f'}{qf} \tag{23}$$

Using [21] and [22], the numerator of the expression [16] for r becomes (dropping the subscript 2)

$$[24] \quad N \equiv Q_0^2 FG + \bar{F}\bar{G} = (2Q)^{-1}[(Q_0^2 - Q^2)\sin\Delta\phi - \Gamma Q^2 \cos\Delta\phi + \tfrac{1}{4}\Gamma^2 Q^2 \sin\Delta\phi]$$

where $\Delta\phi$ is the phase increment across the stratification of thickness $\Delta z = z_2 - z_1$:

$$[25] \quad \Delta\phi = \int_{-\Delta z/2}^{\Delta z/2} dz\, q(z) = 2\int_0^{\Delta z/2} dz\, q(z)$$

The zeros of N are the zeros of r. When $\Gamma = 0$ at the boundary, the Liouville–Green functions make r proportional to $(Q_0^2 - Q^2)\sin\Delta\phi$, and thus there are two kinds of zero: (*i*) a Brewster or Green angle θ_0 satisfying $Q_0^2 = Q^2$, which gives

$$[26] \quad \tan^2\theta_0 = \frac{(k/k_0)^2 - (f/f_0)^2}{1 - (k/k_0)^2}$$

and (*ii*) an interference zero given by $\Delta\phi = n\pi$ where n is an integer. (Note that although the Brewster and Green angle formulae [9] and [10] are special cases of [26], the situation under consideration here is that of a symmetric layer immersed in a uniform medium. The reason for the correspondence with the formulae of Sect. 2, which apply to reflection at a sharp boundary between two uniform media, is that if there is Brewster–Green transparency at the first boundary, there will also be transparency at the second boundary, because of the symmetry of the stratification.)

When Γ is nonzero the distinction between the two kinds of zero disappears. The zeros of r occur when (from [24])

$$[27] \quad \{Q_0^2 - Q^2(1 - \Gamma^2/4)\}\sin\Delta\phi = \Gamma Q^2 \cos\Delta\phi$$

This approximate solution of the $r = 0$ eigenvalue problem will be compared with exact solutions for a family of symmetric stratifications in the next section.

4. Locii of zero r: An example

An illustration of the above results will be provided by a simple symmetric stratification. For concreteness we will discuss the reflection of electromagnetic waves by the family of dielectric function profiles (9)

$$[28] \quad \varepsilon(z) = \begin{cases} \varepsilon_0, & |z| > \Delta z/2 \\ \varepsilon_u + \Delta\varepsilon[1 - (2z/\Delta z)^2], & |z| < \Delta z/2 \end{cases}$$

(we take the magnetic permeability μ to be constant.) When $\Delta\varepsilon = 0$ this is a uniform layer. Then with

$$[29] \quad F = \cos qz, \qquad G = \sin qz, \qquad q^2 = \frac{\omega^2}{c^2}(\varepsilon_u - \varepsilon_0 \sin^2\theta)$$

the formula [16] gives, for the s and p polarizations,

$$[30] \quad r_s = e^{-iq_0\Delta z}\frac{(q_0^2 - q^2)\sin(q\Delta z)}{(q_0^2 + q^2)\sin(q\Delta z) + 2iq_0 q\cos(q\Delta z)}$$

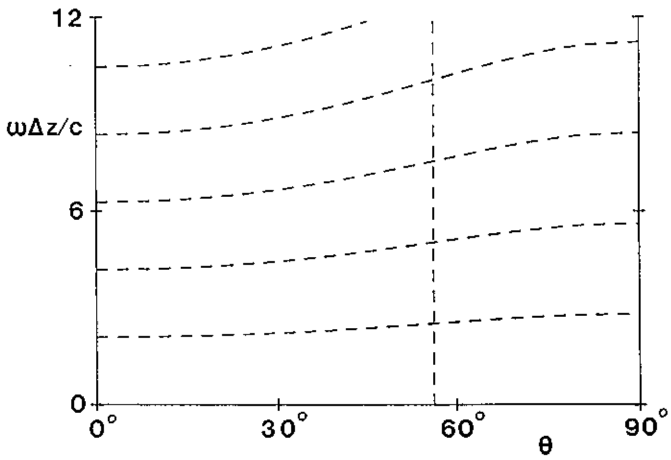

FIG. 2. Locii of zero reflectance for a uniform layer of thickness Δz. The curves apply to both s and p polarizations. The vertical line at the Brewster angle (56.3°) applies to the p polarization only.

$$[31] \quad r_p = e^{-iq_0\Delta z}\frac{(Q_0^2 - Q^2)\sin(q\Delta z)}{(Q_0^2 + Q^2)\sin(q\Delta z) + 2iQ_0 Q\cos(q\Delta z)}$$

Reflection is zero for both polarizations whenever $q\Delta z = n\pi$ (n = integer), that is, at angles of incidence and thicknesses satisfying

$$[32] \quad \left(\frac{\omega}{c}\Delta z\right)^2(\varepsilon_u - \varepsilon_0\sin^2\theta) = n^2\pi^2$$

In addition, the p polarization has zero reflection at the Brewster angle $\theta_B = \arctan(\varepsilon_u/\varepsilon_0)^{1/2}$. Locii of zero reflection in the angle of incidence, thickness plane are shown in Fig. 2. The dielectric function values used there and in the following are $\varepsilon_0 = 1$, $\varepsilon_u = (3/2)^2$, corresponding to glass in air.

The $\Delta\varepsilon = 0$ case (uniform layer) has two separate kinds of zero: an interference zero and a Brewster zero. (At isolated points, where the interference and Brewster locii cross, there are double zeros.) This distinction is lost when $\Delta\varepsilon \neq 0$: the locii now change continuously from interferencelike curves at small and large angles of incidence, to Brewster-like curves near the Brewster angle. Thus a reflection zero at a given angle and thickness may be thought of as primarily being due to interference or to equality of effective wave number components, but there is always an admixture of both. This is seen in Figs. 3 and 4, where zero-reflectance locii are shown for $\Delta\varepsilon = +0.3$ and -0.3, respectively (these values correspond to refractive indices of 1.6 and 1.4 at the centre of the layer).

It is interesting to compare these locii with those obtained from the approximate equality [27] based on the Liouville–Green wave functions. For the profile [28], the phase increment across the nonuniform layer can be evaluated analytically. We find, for $\Delta\varepsilon > 0$,

$$[33] \quad \Delta\phi = \frac{\omega}{c}\Delta z(\Delta\varepsilon)^{1/2}\{[\alpha - \alpha^2]^{1/2} + \arcsin(\alpha^{1/2})\}/2\alpha$$

where $\alpha = \Delta\varepsilon/(\varepsilon_u + \Delta\varepsilon - \varepsilon_0\sin^2\theta)$. For $\Delta\varepsilon < 0$ the result is

$$[34] \quad \Delta\phi = \frac{\omega}{c}\Delta z|\Delta\varepsilon|^{1/2} \times \left\{[\alpha + \alpha^2]^{1/2} + \log\frac{\alpha^{1/2} + [1+\alpha]^{1/2} - 1}{\alpha^{1/2} - [1+\alpha]^{1/2} + 1}\right\}/2\alpha$$

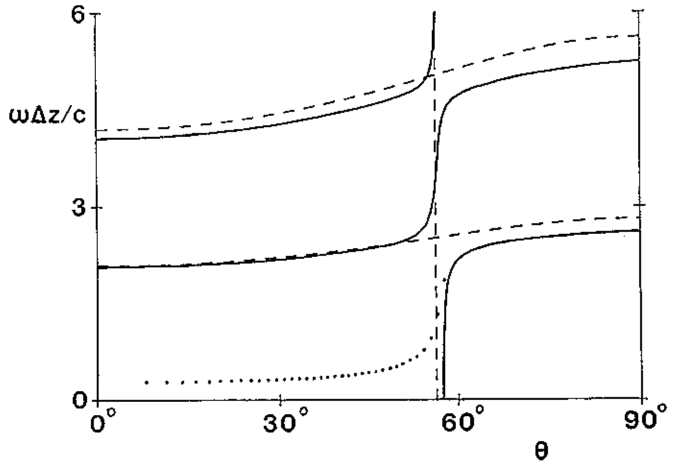

FIG. 3. Zero-reflectance locii (p polarization) for the symmetric stratification [28], with $\varepsilon_0 = 1$, $\varepsilon_u = (3/2)^2$, $\Delta\varepsilon = +0.3$ (— curves). The – – – curves are for the uniform layer ($\Delta\varepsilon = 0$), which were shown in Fig. 2 (these are reproduced here for comparison only). Also shown is a short-wave approximation (· · ·) zero-reflectance locus [27]. Deviation from the exact locii is visible (on this scale) only for the lowest locus, at small $\omega\Delta z/c$.

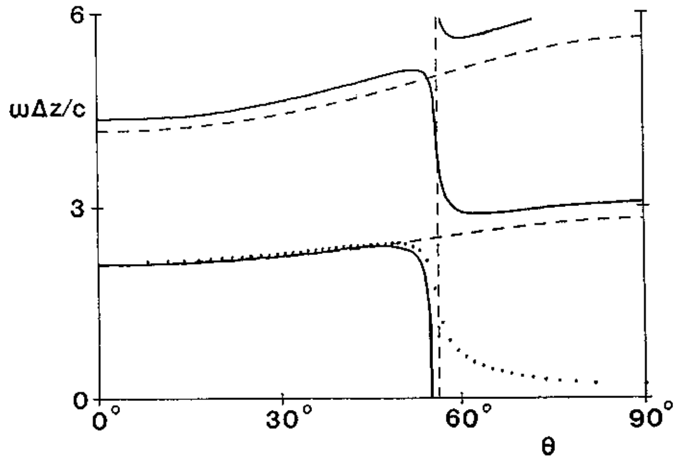

FIG. 4. As for Fig. 3, with $\Delta\varepsilon = -0.3$. The exact zero-reflectance locii in both figures have been plotted using the $L3$ algorithm of ref. 10.

where now $\alpha = |\Delta\varepsilon|/(\varepsilon_u + \Delta\varepsilon - \varepsilon_0 \sin^2\theta)$. The dimensionless quantity Γ is

$$[35] \qquad \Gamma = \frac{2\Delta\varepsilon}{q_u \Delta z}\left\{\frac{2}{\varepsilon_u} - \frac{1}{\varepsilon_u - \varepsilon_0 \sin^2\theta}\right\}$$

The short-wavelength (high-frequency) approximations are compared with the exact results in Figs. 3 and 4. The agreement is so good that for most of the locii there is no visible deviation (on the scale used). However, at small values of $\omega\Delta z/c$ the approximations fail as expected.

5. Conclusions

We will now summarize and comment on the main results of this paper:

(*i*) Symmetric stratifications are likely candidates in the search for zero reflection; and conversely, general profiles may show small reflectance, but only rarely exactly no reflectance. This is related to the analytic structure of the reflection amplitude r.

(*ii*) For uniform media, zero reflection can be classified into a Malus–Brewster–Green type, arising from equality of effective wave number component q/f, and an interference type. This distinction is lost in nonuniform media, as was seen in the example considered (Figs. 3 and 4).

(*iii*) Locii of zero reflectance do not cross, in the general case, and the passage from the uniform to the nonuniform case is singular: a small change in parameter (in our example this was $\Delta\varepsilon$) will switch from one kind of curve to a very different kind. The singularity occurs at the point where there was a crossing of the zero-r locii in the uniform case.

(*iv*) Short-wave approximations, based on the Liouville–Green approximate wave functions, give excellent results for large and even moderate thickness to wavelength ratios ($\omega\Delta z/c \geq 1$). The high-frequency results fail at small $\omega\Delta z/c$, as they must, since they depend on the smallness of the dimensionless quantity Γ [35], which diverges as $\omega\Delta z/c$ tends to zero. The short-wave approximation has the further advantage of giving an explicit transcendental equation for zero reflectance, [27]. When $\Gamma = 0$ this factorizes into the uniform layer equations $Q_0^2 = Q^2$ and $\sin \Delta\phi = 0$, but for $\Gamma \neq 0$ (no matter how small) the zero-r locii are given by a single equation, and thus no crossing of locii occurs.

(*v*) Multiple zeros (for a given thickness, the existence of two or more angles of incidence at which there is no reflection) are of practical importance. Because of the "switching" of locii as $\Delta\varepsilon$ changes sign, two nearly uniform layers with different impurity dopings can have very different properties. With the help of zero-reflectance locii such as displayed here, one can design stratifications that do not reflect at a particular angle of incidence. With double zeros, small reflectance over a large range of angles is possible.

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