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Invariants of electromagnetic beams

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Abstract
Seven quantities are found to be invariant for a monochromatic electromagnetic beam (that is, they have the same value everywhere along the length of the beam). These quantities are related to the conservation of energy, momentum and angular momentum. The simplest of them can be interpreted as the momentum content per unit length of the beam, P'_z. The energy content per unit length U' depends on position, in general, and we always have $U' \geqslant cP'_z$. The angular momentum content per unit length J'_z is also not an invariant. The six other invariants are integrals over elements of the Maxwell stress tensor. Examples of the invariants, and of the non-invariant quantities U' and J'_z, are given for TM and 'circularly polarized' beams based on exact solutions of the Maxwell equations.

Keywords: light beams, laser beams, integral invariants

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Van Enk and Nienhuis [1] and Barnett and Allen [2] introduced the concepts of cycle-averaged energy, momentum and angular momentum content per unit length of a monochromatic light beam:

$$U' = \int d^2r\, \bar{u}, \qquad \boldsymbol{P}' = \int d^2r\, \bar{\boldsymbol{p}}, \qquad \boldsymbol{J}' = \int d^2r\, \bar{\boldsymbol{j}}. \tag{1}$$

Here $\int d^2r$ stands for $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy = \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\phi$ and u, $\boldsymbol{p}$ and $\boldsymbol{j}$ are the instantaneous energy, momentum and angular momentum densities [3]:

$$u(\boldsymbol{r}, t) = \frac{1}{8\pi}(E^2 + B^2), \qquad \boldsymbol{p}(\boldsymbol{r}, t) = \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B}, \qquad \boldsymbol{j}(\boldsymbol{r}, t) = \boldsymbol{r}\times\boldsymbol{p}. \tag{2}$$

We use the notation U' since $dU = U'\, dz$ is the energy content in a thin slice of thickness dz of the beam (which we assume has net propagation in the z direction), so that U' may be viewed as dU/dz.

In (2) the electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are real. For monochromatic beams of angular frequency ω it is convenient to write (for example)

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}\} = \tfrac{1}{2}\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t} + \boldsymbol{E}^*(\boldsymbol{r})e^{i\omega t}\} \tag{3}$$

and then in terms of the complex fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ we have the cycle-averaged quantities

$$\begin{aligned}\bar{u} &= \frac{1}{16\pi}[\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{E}^*(\boldsymbol{r}) + \boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{B}^*(\boldsymbol{r})]\\ \bar{\boldsymbol{p}} &= \frac{1}{16\pi c}[\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{B}^*(\boldsymbol{r}) + \boldsymbol{E}^*(\boldsymbol{r})\times\boldsymbol{B}(\boldsymbol{r})].\end{aligned} \tag{4}$$

In the case of *steady beams*, for which the complex fields are related by $\boldsymbol{E} = \pm i\boldsymbol{B}$ [4, 5], the energy and momentum densities are time-independent:

$$u = \frac{1}{8\pi}|\boldsymbol{E}|^2 = \frac{1}{8\pi}|\boldsymbol{B}|^2, \qquad c\boldsymbol{p} = \frac{i}{8\pi}\boldsymbol{E}\times\boldsymbol{E}^* = \frac{i}{8\pi}\boldsymbol{B}\times\boldsymbol{B}^*. \tag{5}$$

Likewise all elements of the Maxwell stress tensor are time-independent when $\boldsymbol{E} = \pm i\boldsymbol{B}$ and thus the results to be derived apply to steady beams at all instants of time.

One might think that U', $\boldsymbol{P}'$ and $\boldsymbol{J}'$ would all be invariants, i.e. that the energy, momentum and angular momentum contents per unit length of the beam should not vary along the beam. But a counter-example to U' being independent of z is known: in section 5 of [4] the energy per unit length was evaluated exactly for a TM beam based on the wavefunction ψ_{10} defined in (34) and it was found to depend on z. Likewise we shall give a counter-example to the invariance of J'_z. However, P'_z and six more quantities associated with components of the Maxwell stress tensor are invariants for electromagnetic beams, as we shall now show.

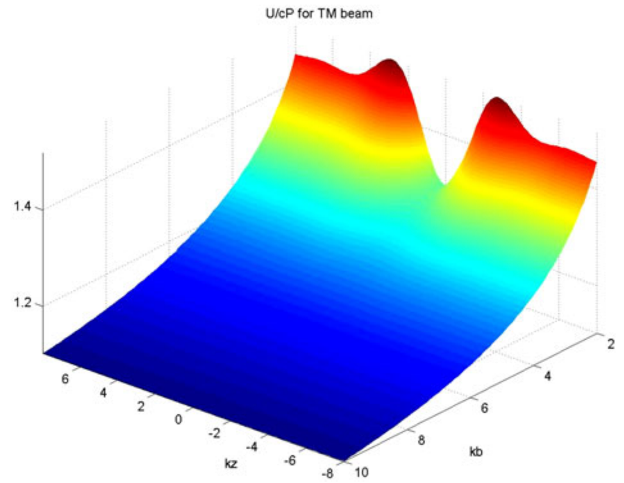

Figure 1. The ratio of the energy content to c times momentum content per unit length, U'/cP_z', of a TM beam constructed from the wavefunction ψ_{10}. The deviation from unity is greatest for small $\beta = kb$, i.e. for highly focused beams. The value of the smallest β shown ($\beta = 2$) corresponds to a beam converging onto and diverging from the focal region at a cone half-angle of 45°. P_z' is an invariant (independent of z) and U' is dependent on z.

2. Conservation of energy

The conservation of energy for an electromagnetic field in free space reads [3]

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} = 0 \tag{6}$$

where $\boldsymbol{S} = c^2\boldsymbol{p}$ is the energy flux density. The cycle average of (6) gives

$$0 = \nabla \cdot \bar{\boldsymbol{p}} = \partial_x \bar{p}_x + \partial_y \bar{p}_y + \partial_z \bar{p}_z. \tag{7}$$

Let us apply $\int \mathrm{d}^2 r$ to (7). The $\partial_x \bar{p}_x$ and $\partial_y \bar{p}_y$ terms integrate to zero: for example, $\int_{-\infty}^{\infty} \mathrm{d}x\, \partial_x \bar{p}_x = \bar{p}_x \big]_{-\infty}^{\infty} = 0$. Thus we are left with

$$\partial_z \int \mathrm{d}^2 r\, \bar{p}_z = 0, \qquad \text{i.e. } P_z' = \int \mathrm{d}^2 r\, \bar{p}_z = \text{constant}. \tag{8}$$

Thus the momentum content per unit length, along the direction of net propagation of the beam, is an invariant. But note that this invariance follows from *energy* conservation and that momentum enters through the relation $\boldsymbol{S} = c^2\boldsymbol{p}$ between the energy flux density and the momentum density.

We now show that $U' \geqslant cP_z'$. We have, using real fields,

$$\begin{aligned} u - cp_z &= \frac{1}{8\pi}[E^2 + B^2 - 2(\boldsymbol{E} \times \boldsymbol{B})_z] \\ &= \frac{1}{8\pi}[(E_x - B_y)^2 + (E_y + B_x)^2 + E_z^2 + B_z^2] \geqslant 0 \end{aligned} \tag{9}$$

$U' - cP_z'$ is the integral over an xy (transverse) plane of the right-hand side. It is non-negative and can only be zero if $E_x = B_y$, $E_y = -B_x$ and $E_z = 0 = B_z$ everywhere. But pure TEM transversely bounded beams do not exist (theorem 2.1 of section 2 in [6]), so $u = cp_z$ cannot hold everywhere. Thus the energy content per unit length is, in general, greater than c times the momentum content. Figure 1 shows the ratio U'/cP_z' for a TM beam based on the wavefunction ψ_{10} defined in (34). The elevation gives the ratio of equation (54) to equation (53) in [4]; the latter invariant is reproduced in (29), but the former is a complicated function of z and is not reproduced in this paper.

3. Conservation of momentum

The rate of change of field momentum can be written in terms of the Maxwell stress tensor (see, for example, section 6.8 of [3]). We shall reverse its sign so that momentum conservation can be written in a form analogous to (6), as has been done by Barnett [7]:

$$\tau_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}(E^2 + B^2)\delta_{ij} - E_i E_j - B_i B_j\right]. \tag{10}$$

In the absence of charges and currents the rate of change of the momentum density is given by

$$\frac{\partial p_i}{\partial t} + \sum_j \partial_j \tau_{ij} = 0. \tag{11}$$

(Comparison with (6) shows that τ_{ij} can be regarded as the momentum flux density [7].) Averaging over an integral number of cycles gives us $\sum_j \partial_j \bar{\tau}_{ij} = 0$. When we integrate these three equations over the xy plane, the x and y derivatives give zero as before, so we are left with

$$\partial_z \int \mathrm{d}^2 r\, \bar{\tau}_{iz} = 0, \qquad i = x, y, z. \tag{12}$$

Thus the invariance of three quantities follows from momentum conservation:

$$\begin{aligned} T_{xz}' &= \int \mathrm{d}^2 r\, \bar{\tau}_{xz} = -\frac{1}{4\pi}\int \mathrm{d}^2 r\, [\overline{E_x E_z} + \overline{B_x B_z}] \\ T_{yz}' &= \int \mathrm{d}^2 r\, \bar{\tau}_{yz} = -\frac{1}{4\pi}\int \mathrm{d}^2 r\, [\overline{E_y E_z} + \overline{B_y B_z}] \\ T_{zz}' &= \int \mathrm{d}^2 r\, \bar{\tau}_{zz} \\ &= \frac{1}{8\pi}\int \mathrm{d}^2 r\, [\overline{E_x^2 + E_y^2 - E_z^2} + \overline{B_x^2 + B_y^2 - B_z^2}]. \end{aligned} \tag{13}$$

(The bars denote cycle-averaging, as elsewhere in this paper. If we wish to use complex fields, as in (3), $\overline{E_x E_z}$ is to be replaced by $\frac{1}{2}\,\mathrm{Re}(E_x E_z^*)$, for example.) These three invariants have the dimension of U', i.e. of energy/length.

The T_{xz}' and T_{yz}' invariants will often be zero by symmetry. For example, if (as will always be the case here) the z axis ($x = 0 = y$) is taken as the beam axis, the TM beam derived from the vector potential $\boldsymbol{A} = [0, 0, \psi]$, with ψ independent of azimuthal angle ϕ, will have [4]

$$\begin{aligned} \boldsymbol{B} &= [\sin\phi, -\cos\phi, 0]\partial_\rho \psi, \\ \boldsymbol{E} &= \frac{\mathrm{i}}{k}[\cos\phi\,\partial_\rho\partial_z, \sin\phi\,\partial_\rho\partial_z, \partial_z^2 + k^2]\psi. \end{aligned} \tag{14}$$

(Here, and elsewhere in this paper, $k = \omega/c$.) Thus the transverse components of $\boldsymbol{B}$ and $\boldsymbol{E}$ are proportional to $\sin\phi$

or $\cos\phi$ and integrate to zero over ϕ. This is also true if ψ has $e^{im\phi}$ dependence: we find

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A} = [\partial_y, -\partial_x, 0]\psi \\ &= \left[\sin\phi\,\partial_\rho + \frac{\mathrm{i}m}{\rho}\cos\phi, -\cos\phi\,\partial_\rho + \frac{\mathrm{i}m}{\rho}\sin\phi, 0\right]\psi \end{aligned} \tag{15}$$

$$\begin{aligned}\boldsymbol{E} &= \frac{\mathrm{i}}{k}[\nabla(\nabla\cdot\boldsymbol{A}) + k^2\boldsymbol{A}] = \frac{\mathrm{i}}{k}[\partial_x\partial_z, \partial_y\partial_z, \partial_z^2 + k^2]\psi \\ &= \frac{\mathrm{i}}{k}\left[\cos\phi\,\partial_\rho\partial_z - \frac{\mathrm{i}m}{\rho}\sin\phi\,\partial_z, \sin\phi\,\partial_\rho\partial_z \right. \\ &\quad \left. + \frac{\mathrm{i}m}{\rho}\cos\phi\,\partial_z, \partial_z^2 + k^2\right]\psi \end{aligned} \tag{16}$$

on using $\partial_x = \cos\phi\,\partial_\rho - \rho^{-1}\sin\phi\,\partial_\phi$, $\partial_y = \sin\phi\,\partial_\rho + \rho^{-1}\cos\phi\,\partial_\phi$. Thus $E_x E_z^*$ will still be first order in $\sin\phi$ and $\cos\phi$ and again integration over the azimuthal angle will give zero T'_{xz} and T'_{yz}.

4. Conservation of angular momentum

Just as the rate of change of electromagnetic momentum density is given by the spatial derivatives of the Maxwell stress tensor in (11), the rate of change of electromagnetic angular momentum density is given by spatial derivatives of an angular momentum flux density tensor, defined by Barnett [7] (see also problem 6.11 of [3]). This is constructed from the momentum flux density tensor τ_{ij} defined in (10):

$$\mu_{\ell i} = \sum_j\sum_k \varepsilon_{ijk} x_j \tau_{k\ell} \tag{17}$$

and the rate of change of the angular momentum density $\boldsymbol{j}$ is given by [7]

$$\frac{\partial j_i}{\partial t} + \sum_\ell \partial_\ell \mu_{\ell i} = 0. \tag{18}$$

For monochromatic beams, averaging over one cycle gives $\sum_\ell \partial_\ell \bar{\mu}_{\ell i} = 0$. We again integrate these three equations over an xy plane. The x and y derivatives integrate to zero for finite beams. What remains is

$$\partial_z \int \mathrm{d}^2 r\,\bar{\mu}_{zi} = 0, \qquad i = x, y, z. \tag{19}$$

Thus three invariants result from the conservation of angular momentum:

$$\begin{aligned} M'_{zx} &= \int \mathrm{d}^2 r\,\bar{\mu}_{zx} = \int \mathrm{d}^2 r\,[y\bar{\tau}_{zz} - z\bar{\tau}_{yz}] \\ M'_{zy} &= \int \mathrm{d}^2 r\,\bar{\mu}_{zy} = \int \mathrm{d}^2 r\,[z\bar{\tau}_{xz} - x\bar{\tau}_{zz}] \\ M'_{zz} &= \int \mathrm{d}^2 r\,\bar{\mu}_{zz} = \int \mathrm{d}^2 r\,[x\bar{\tau}_{yz} - y\bar{\tau}_{xz}]. \end{aligned} \tag{20}$$

The elements of the momentum flux density tensor appearing here are the same as those made explicit in the momentum conservation invariants (13).

5. The invariants of azimuthally symmetric TM beams

Let ψ be any solution of the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$. Then monochromatic fields (with angular frequency $\omega = ck$) satisfying Maxwell's equations in free space have the spatial parts $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, $\boldsymbol{E} = \frac{\mathrm{i}}{k}[\nabla(\nabla\cdot\boldsymbol{A}) + k^2\boldsymbol{A}]$. When $\boldsymbol{A} = [0, 0, \psi]$ and ψ is independent of the azimuthal angle ϕ, $\boldsymbol{B}$ and $\boldsymbol{E}$ are given by (14). The resulting z component of the cycle-averaged momentum density is, from (4) and (14),

$$\bar{p}_z = \frac{1}{8\pi ck}\,\mathrm{Im}\{(\partial_\rho\psi^*)(\partial_\rho\partial_z\psi)\}. \tag{21}$$

The integral $P'_z = \int \mathrm{d}^2 r\,\bar{p}_z = 2\pi\int_0^\infty \mathrm{d}\rho\,\rho\bar{p}_z$ gives our first beam invariant (8). The other invariants are integrals over three cycle-averaged elements of the momentum flux density tensor, which we express in terms of complex electric and magnetic fields:

$$\begin{aligned} \bar{\tau}_{xz} &= -\frac{1}{8\pi}\,\mathrm{Re}(E_x E_z^* + B_x B_z^*) \\ \bar{\tau}_{yz} &= -\frac{1}{8\pi}\,\mathrm{Re}(E_y E_z^* + B_y B_z^*) \\ \bar{\tau}_{zz} &= \frac{1}{16\pi}\{|E_x|^2 + |E_y|^2 - |E_z|^2 + |B_x|^2 + |B_y|^2 - |B_z|^2\}. \end{aligned} \tag{22}$$

For the azimuthally symmetric TM beam, with complex $\boldsymbol{E}$ and $\boldsymbol{B}$ given by (14), we saw in section 3 that $\bar{\tau}_{xz}$ and $\bar{\tau}_{yz}$ are proportional to $\cos\phi$ and $\sin\phi$, respectively, and thus give zero on integration over ϕ. The zz component is independent of ϕ:

$$\bar{\tau}_{zz} = \frac{1}{16\pi}\{k^{-2}[|\partial_\rho\partial_z\psi|^2 - |(\partial_z^2 + k^2)\psi|^2] + |\partial_\rho\psi|^2\}. \tag{23}$$

Because of the ϕ dependence of the elements of the momentum flux density tensor, there are only two non-zero invariants of azimuthally symmetric TM beams, namely

$$P'_z = \frac{1}{4ck}\int_0^\infty \mathrm{d}\rho\,\rho\,\mathrm{Im}\{(\partial_\rho\psi^*)(\partial_\rho\partial_z\psi)\} \tag{24}$$

$$T'_{zz} = \frac{1}{8k^2}\int_0^\infty \mathrm{d}\rho\,\rho\{[|\partial_\rho\partial_z\psi|^2 - |(\partial_z^2 + k^2)\psi|^2] + k^2|\partial_\rho\psi|^2\}. \tag{25}$$

For comparison, we also give here the expression for the non-invariant energy per unit length U' of an azimuthally symmetric TM beam:

$$U' = \frac{1}{8k^2}\int_0^\infty \mathrm{d}\rho\,\rho\{|\partial_\rho\partial_z\psi|^2 + |(\partial_z^2 + k^2)\psi|^2 + k^2|\partial_\rho\psi|^2\}. \tag{26}$$

We note that $U' - cP'_z$ is positive, in accord with (9):

$$U' - cP'_z = \frac{1}{8k^2}\int_0^\infty \mathrm{d}\rho\,\rho\{|\partial_\rho\partial_z\psi + \mathrm{i}k\partial_\rho\psi|^2 + |(\partial_z^2 + k^2)\psi|^2\}. \tag{27}$$

The z component of the angular momentum density is $j_z = \rho p_\phi$, where $p_\phi = -p_x\sin\phi + p_y\cos\phi$ is the azimuthal component of the momentum density. From (14) we find that p_ϕ is identically zero for TM beams with ψ independent of ϕ, so j_z is zero everywhere: there cannot be an azimuthal component of $\boldsymbol{E}\times\boldsymbol{B}$ when one of the fields ($\boldsymbol{B}$ in this case) is purely azimuthal.

As mentioned at the end of section 2, P'_z and U' were evaluated analytically for a particular beam wavefunction in [4] and figure 1 shows the ratio U'/cP'_z for this wavefunction, namely ψ_{10} defined in (34). For this wavefunction, which is chosen from a set generalizing the Sheppard and Saghafi solution $(\sin kR)/kR$ [8, 9] and is the simplest of the set giving a non-divergent energy per unit length [4], we find

$$T'_{zz} = \frac{A_0^2}{16\beta^6}\{8\beta^4 C - 28\beta^3 S + 54\beta^2 C - 60\beta S + 30C + 6\beta^2 - 30\} \tag{28}$$

where $\beta = kb$ and $C = \cosh 2\beta$, $S = \sinh 2\beta$ (A_0^2 is a constant of dimension energy/length). For comparison, the momentum per unit length is (from equation (53) of [4]) given by

$$cP'_z = \frac{A_0^2}{16\beta^5}\{2\beta^3 S - 5\beta^2 C + 6\beta S - 3C - \beta^2 + 3\}. \tag{29}$$

(The energy per unit length U' is more complicated, depending on z; it is given in equation (54) of [4].) The apparent exponential growth with β arises from the wavefunction: if we were to normalize $|\psi|$ to unity at the origin in the focal plane ($\rho = 0,\ z = 0$), the above expressions would be divided by $((\sinh\beta/\beta^2) - (\cosh\beta/\beta))^2$ and the leading terms in T'_{zz} and cP'_z at large β would be A_0^2 and $A_0^2/4$, respectively.

6. The invariants of 'CP' beams

We saw in the preceding section that TM beams based on solutions of the Helmholtz equation which are independent of ϕ will all have identically zero angular momentum density, so that all of the M' invariants are automatically zero, since these originate in the conservation of angular momentum. We now examine 'circularly polarized' ('CP') beams. Theorem 2.3 of [6] states that 'beams which are everywhere circularly polarized in a fixed plane do not exist'; hence the quotes around 'CP' (the textbook circularly polarized 'beam' is not a finite beam but an infinite plane wave). However, it is possible to construct a *steady beam* (i.e. one in which the complex fields satisfy $\boldsymbol{E} = \pm i\boldsymbol{B}$) which is circularly polarized in the plane wave (wide beam) limit. This was done in section 4 of [6], where the polarization properties of this 'CP' beam were discussed in detail. Here we just consider the invariants of the beam and the non-invariants U' and J'_z. The vector potential is

$$\boldsymbol{A} = [-(\partial_z + ik), -i(\partial_z + ik), (\partial_x + i\partial_y)]\psi. \tag{30}$$

When ψ is independent of ϕ, $(\partial_x + i\partial_y)\psi = e^{i\phi}\partial_\rho\psi$ and the energy density is found from (5) and equation (37) of [6] to be, on using $(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 + k^2)\psi = 0$,

$$u = \frac{1}{8\pi}\{k^2|(\partial_z + ik)\psi|^2 + |(\partial_z + ik)\partial_z\psi|^2 + |(\partial_z + ik)\partial_\rho\psi|^2\}. \tag{31}$$

Likewise from (5) and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, the z components of the momentum density and the angular momentum density $j_z = xp_y - yp_x = \rho p_\phi$ are given by

$$cp_z = \frac{1}{8\pi}\left\{k^2|(\partial_z + ik)\psi|^2 + |(\partial_z + ik)\partial_z\psi|^2 - |\partial_\rho^2\psi|^2 - \frac{1}{\rho^2}|\partial_\rho\psi|^2\right\} \tag{32}$$

$$cj_z = -\frac{\rho}{4\pi}\,\mathrm{Im}\{[(\partial_\rho^2 + ik\partial_z + k^2)\psi^*][(\partial_z + ik)\partial_\rho\psi]\}. \tag{33}$$

It is clear from (31) and (32) that $u - cp_z \geqslant 0$, so that $U' \geqslant cP'_z$, in accord with the general result of section 2.

We shall again use the simplest of the $\psi_{\ell m} = j_\ell(kR)P_{\ell m}(\frac{z-ib}{R})$ set of exact solutions of the Helmholtz equation [8, 9] which gives convergent integrals for U' [4] and J'_z, namely

$$\psi_{10} = \frac{A_0}{k}\left[\frac{\sin kR}{(kR)^2} - \frac{\cos kR}{kR}\right]\frac{z - ib}{R}, \qquad R^2 = \rho^2 + (z - ib)^2. \tag{34}$$

(The factor k^{-1} has been inserted to give A_0 the dimension of vector potential: see (30).) The diffraction length b (also known as the Rayleigh length) gives the length of the focal region of the beam and also determines the beam waist, which is $(2b/k)^{1/2}$. It is convenient in performing the differentiations and integrations to transform to oblate spheroidal coordinates ξ, η:

$$\rho = b\sqrt{(1 + \xi^2)(1 - \eta^2)}, \qquad z = b\xi\eta, \qquad R = b(\xi - i\eta). \tag{35}$$

The derivatives are written out explicitly in the appendix of [4]; in integrating over a plane of fixed z we also make use of (A.7) of [4], namely

$$\int_0^\infty d\rho\,\rho p_z = b^2\int_0^1 d\eta\,\eta^{-1}[\eta^2 + (\zeta/\eta)^2]p_z, \qquad \zeta = z/b. \tag{36}$$

The results for the momentum, energy and angular momentum content per unit length of the ψ_{10} 'CP' beam are as follows:

$$cP'_z = \frac{A_0^2}{16\beta^5}\{4\beta^4(C + S) - 6\beta^3(C + S) + \beta^2(6C + 7S) - 3\beta(C + 2S) + 3C - \beta^2 + 3\beta - 3\} \tag{37}$$

$$U' = \frac{A_0^2}{32\beta^6}\left\{8\beta^5(C + S) - 8\beta^4(C + S) + 2\beta^3(2C + 3S) - 3\beta^2 C + 2\beta^4 - 4\beta^3 + 3\beta^2 + \frac{1}{\zeta^6}[2(\beta\zeta)^4 + 6(\beta\zeta)^2 + 15 + 2\beta\zeta(4(\beta\zeta)^2 - 15)\sin 2\beta\zeta + 3(8(\beta\zeta)^2 - 5) \times \cos 2\beta\zeta]\right\} \tag{38}$$

$$\omega J'_z = \frac{A_0^2}{32\beta^5}\left\{8\beta^4(C + S) - 4\beta^3(C + S) - 2\beta^2 C + 6\beta S - 3C - 2\beta^4 + 4\beta^3 - 4\beta^2 + 3 + \frac{1}{\zeta^6}[2\zeta^4(\beta^4 + 2\beta^3) + 6\zeta^2(\beta^2 + \beta) + 15 + (8(\beta\zeta)^3 - 12\beta^2\zeta^3 - 30\beta\zeta) \times \sin 2\beta\zeta + (8\beta^3\zeta^4 + 24(\beta\zeta)^2 - 6\beta\zeta^2 - 15) \times \cos 2\beta\zeta]\right\}. \tag{39}$$

In these formulae $\beta = kb$, $\zeta = z/b$ (so $\beta\zeta = kz$), $C = \cosh 2\beta$ and $S = \sinh 2\beta$. Note that P'_z is an invariant, independent of z, while U' and J'_z depend on z. The non-invariance of J'_z provides the counter-example promised in section 1.

The z-dependent parts of U' and J'_z are not singular at $z = 0$ (the focal plane) and are even in z, as expected. In the limit of large β we have

$$U' \to cP'_z, \qquad J'_z \to U'/\omega. \tag{40}$$

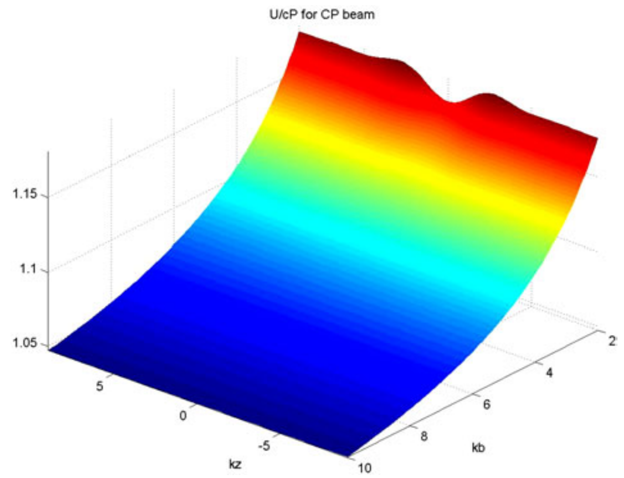

Figure 2. U'/cP'_z for the ψ_{10} 'CP' beam. The deviation from unity is most marked for tightly focused beams (small $\beta = kb$). For large β the ratio of energy to c times momentum content per unit length tends to $1 + (2\beta)^{-1} + O(\beta^{-2})$.

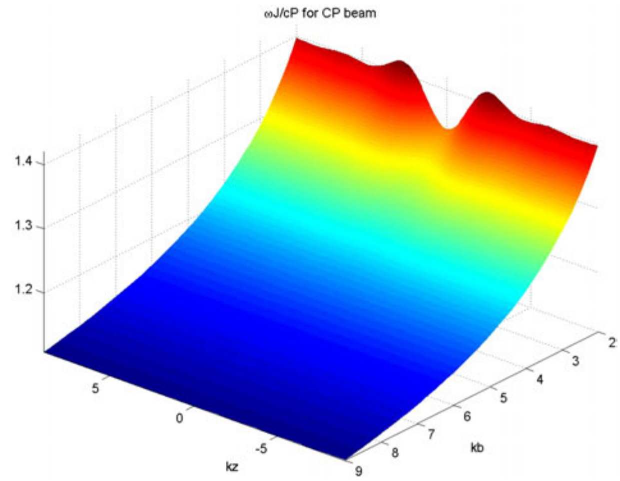

Figure 3. The non-invariant $\omega J'_z$ divided by the invariant cP'_z for the 'CP' beam constructed from ψ_{10}. For large $\beta = kb$ the ratio of ω times angular momentum content to c times momentum content per unit length tends to unity as $1 + \beta^{-1} + O(\beta^{-2})$.

Thus a broad, weakly focused circularly polarized beam can be thought of as consisting of Einstein photons, each carrying energy $\hbar\omega$, momentum $\hbar\omega/c$ and angular momentum $\hbar$.

Figures 2 and 3 show U'/cP'_z and $\omega J'_z/cP'_z$ for the 'CP' beam with $\psi = \psi_{10}$. As for the TM beam, the derivations from unity are most marked for small $\beta = kb$ for tightly focused beams.

We now look at the invariants arising out of the conservation of momentum, given by (13). The 'CP' beam is *steady* ($\boldsymbol{E} = \mathrm{i}\boldsymbol{B}$ by construction), so cycle-averaging is not necessary. For example, in (13) we have, with $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}^r \cos\omega t + \boldsymbol{E}^i \sin\omega t$, etc, and $\boldsymbol{E}^r + \mathrm{i}\boldsymbol{E}^i = \mathrm{i}(\boldsymbol{B}^r + \mathrm{i}\boldsymbol{B}^i)$ (so $\boldsymbol{E}^r = -\boldsymbol{B}^i$ and $\boldsymbol{E}^i = \boldsymbol{B}^r$)

$$\begin{aligned}
&E_x(\boldsymbol{r}, t)E_z(\boldsymbol{r}, t) + B_x(\boldsymbol{r}, t)B_z(\boldsymbol{r}, t)\\
&\quad= (E_x^r \cos\omega t + E_x^i \sin\omega t)(E_z^r \cos\omega t + E_z^i \sin\omega t)\\
&\qquad+ (B_x^r \cos\omega t + B_x^i \sin\omega t)(B_z^r \cos\omega t + B_z^i \sin\omega t)\\
&\quad= (-B_x^i \cos\omega t + B_x^r \sin\omega t)(-B_z^i \cos\omega t + B_z^r \sin\omega t)\\
&\qquad+ (B_x^r \cos\omega t + B_x^i \sin\omega t)(B_z^r \cos\omega t + B_z^i \sin\omega t)\\
&\quad= B_x^r B_z^r + B_x^i B_z^i = \mathrm{Re}(B_x B_z^*) \qquad (41)
\end{aligned}$$

(real fields $B_x(\boldsymbol{r}, t)$, etc, on the left, complex fields $B_x(\boldsymbol{r})$ and $B_z(\boldsymbol{r})$ in the final expression). Thus the invariants (13) become, for steady beams,

$$\begin{aligned}
T'_{xz} &= -\frac{1}{4\pi}\int \mathrm{d}^2r\, \mathrm{Re}(B_x B_z^*)\\
T'_{yz} &= -\frac{1}{4\pi}\int \mathrm{d}^2r\, \mathrm{Re}(B_y B_z^*) \qquad (42)\\
T'_{zz} &= \frac{1}{8\pi}\int \mathrm{d}^2r\, [|B_x|^2 + |B_y|^2 - |B_z|^2].
\end{aligned}$$

The magnetic field is $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is given by (30) for 'CP' beams. For ψ independent of the azimuthal angle ϕ we find the complex magnetic field to be [6]

$$\begin{aligned}
B_x &= \mathrm{e}^{\mathrm{i}\phi}(\sin\phi\,\partial_\rho + \mathrm{i}\rho^{-1}\cos\phi)\partial_\rho\psi + \mathrm{i}(\partial_z + \mathrm{i}k)\partial_z\psi\\
B_y &= -\mathrm{e}^{\mathrm{i}\phi}(\cos\phi\,\partial_\rho - \mathrm{i}\rho^{-1}\sin\phi)\partial_\rho\psi - (\partial_z + \mathrm{i}k)\partial_z\psi \qquad (43)\\
B_z &= -\mathrm{i}\mathrm{e}^{\mathrm{i}\phi}(\partial_z + \mathrm{i}k)\partial_\rho\psi.
\end{aligned}$$

We see that (for ψ independent of ϕ) all terms in $B_x B_z^*$ and $B_y B_z^*$ contain $\sin\phi$ or $\cos\phi$ linearly, and thus integrate to zero over ϕ. Thus the invariants T'_{xz} and T'_{yz} are zero for 'CP' beams with $\psi = \psi(\rho, z)$. The integrand of T'_{zz} simplifies (on the use of $(\nabla^2 + k^2)\psi = 0$) to

$$\begin{aligned}
|B_x|^2 + |B_y|^2 - |B_z|^2 &= k^2|(\partial_z + \mathrm{i}k)\psi|^2 + |(\partial_z + \mathrm{i}k)\partial_z\psi|^2\\
&\quad - |(\partial_z + \mathrm{i}k)\partial_\rho\psi|^2 - \frac{2}{\rho}\mathrm{Re}(\partial_\rho^2\psi \cdot \partial_\rho\psi^*). \qquad (44)
\end{aligned}$$

The T'_{zz} invariant for the ψ_{10} 'CP' beam follows from (34) and (44):

$$\begin{aligned}
T'_{zz} &= \frac{A_0^2}{32\beta^6}\{8\beta^5(C + S) - 16\beta^4(C + S) + 2\beta^3(12C + 13S)\\
&\quad - 3\beta^2(11C + 8S) + 6\beta(2C + 5S)\\
&\quad - 15C + 3\beta^2 - 12\beta + 15\}. \qquad (45)
\end{aligned}$$

At large β, T'_{zz} has the same leading term as cP'_z, namely $A_0^2\mathrm{e}^{2\beta}/4\beta$.

It remains for us to evaluate the invariants arising from the conservation of angular momentum, given in (20). Since $8\pi\tau_{zz}$ given in (44) is independent of ϕ when ψ is independent of ϕ, while τ_{xz} and τ_{yz} contain $\cos\phi$ and $\sin\phi$ linearly, integration over ϕ gives zero M'_{zx} and M'_{zy}. Only M'_{zz} can be non-zero for a 'CP' beam with an azimuthally symmetric ψ. The integrand of M'_{zz} is, for a steady beam, $-(4\pi)^{-1}\{x\,\mathrm{Re}(B_y B_z^*) - y\,\mathrm{Re}(B_x B_z^*)\}$. For the 'CP' beam we have, from (43),

$$(xB_y - yB_x)B_z^* = -\mathrm{i}\rho[(\partial_\rho^2 + \partial_z^2 + \mathrm{i}k\partial_z)\psi][(\partial_z - \mathrm{i}k)\partial_\rho\psi^*] \qquad (46)$$

so the integrand of M'_{zz} is

$$\begin{aligned}
&\frac{\rho}{4\pi}\mathrm{Im}\{[(\partial_\rho^2 + \partial_z^2 + \mathrm{i}k\partial_z)\psi][(\partial_z - \mathrm{i}k)\partial_\rho\psi^*]\}\\
&\quad= \frac{\rho}{4\pi}\mathrm{Im}\{[(\mathrm{i}k(\partial_z + \mathrm{i}k) - \rho^{-1}\partial_\rho)\psi][(\partial_z - \mathrm{i}k)\partial_\rho\psi^*]\}. \qquad (47)
\end{aligned}$$

We find

$$\begin{aligned}
\frac{1}{b}M'_{zz} &= -\frac{A_0^2}{32\beta^6}\{8\beta^4(C + S) - 12\beta^3(C + S) + 2\beta^2(7C + 6S)\\
&\quad - 6\beta(C + 2S) + 6C - 2\beta^2 + 6\beta - 6\}. \qquad (48)
\end{aligned}$$

This completes the evaluation of the three non-zero integral invariants for the 'CP' beam based on the ψ_{10} wavefunction. The reader may be interested to learn that any errors (of which the author had made several!) in the formulation and evaluation of the integral invariants lead to expressions which depend on z, similar to those obtained for the non-invariants U' and J'_z.

7. Discussion

It follows from the conservation of energy, momentum and angular momentum that there exist seven beam invariants, each in the form of an integral over a section of the beam: $\int d^2r\, f(\rho, z, \phi) = \int_0^\infty d\rho\, \rho \int_0^{2\pi} d\phi\, f(\rho, z, \phi)$. These integrals are independent of z. They give the flux of energy, and of the components of momentum and angular momentum, through any plane z = constant perpendicular to the beam propagation direction. It is interesting that the energy and angular momentum contents per unit length of the beam are *not* constant in general. For example, it is the total *flux* of energy that is fixed by energy conservation, not the energy content per unit length. The momentum per unit length of the beam *is* constant, but this is because electromagnetic momentum density is the energy flux density divided by c^2, i.e. this result originates in the conservation of energy.

Where there are symmetries, the number of independent non-zero invariants is less than the possible seven. For example, the TM and 'CP' beams based on the ψ_{10} solution of the Helmholtz equation have only two and three non-zero invariants, respectively.

The non-invariants U' and J'_z, giving the energy and angular momentum content per unit length of a beam, are strongly dependent on z only in the focal region of a tightly focused beam. For wide, weakly focused 'CP' beams, we have seen that $U' \approx cP'_z$ and $J'_z \approx U'/\omega$, in accord with the idea of a circularly polarized monochromatic beam consisting of photons of energy $\hbar\omega$, momentum $\hbar\omega/c$ and angular momentum $\hbar$. We always have $U' > cP'_z$, equality being attained only in the limit of an infinite plane wave.

Although the existence of beam invariants is interesting theoretically, one might expect them to be useful as well. Applications are most likely in the field of the manipulation of small particles by laser beams [10–14], since it is the transfer of momentum and angular momentum to these particles that results in the forces and torques that are of interest.

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Note added November 2003: An anonymous referee has noted that the result (8) (and other invariants derived here) can be generalized, for example by tilting the coordinate system, so that the xy plane is no longer perpendicular to the net propagation direction. Parallel transport of these surfaces gives a set of related invariants. All that is required is that the fields vanish at infinity in the new x and y directions.

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