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LETTER TO THE EDITOR

Helical light pulses

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Abstract
Exact solutions of Maxwell's equations for three-dimensionally localized helical pulses are given. These TE (transverse electric) pulses have $e^{\pm i\phi}$ azimuthal dependence. Their energy, momentum and angular momentum are evaluated analytically. For $e^{+i\phi}$ azimuthal dependence the angular momentum is negative, despite the fact that the pulse energy density has a right-handed helical structure.

Keywords: electromagnetic pulses, photons, helicity

(Some figures in this article are in colour only in the electronic version)

There is much interest in the multidimensional states of angular momentum which are in principle available in specially prepared light pulses: for the purposes of communication, of computation and of quantum cryptography. Mair *et al* [1] have demonstrated the entanglement of the orbital angular momentum states of photons. Molina-Terriza *et al* [2] have proposed schemes for the preparation of photons in multidimensional states of orbital angular momentum. Leach *et al* [3, 4] have demonstrated an interferometric method for measuring the orbital angular momentum of single photons.

The analytical approaches used so far have involved the paraxial approximation. In this letter we shall give exact solutions of Maxwell's equations for a set of three-dimensionally localized pulses with a helical structure. The energy, momentum and angular momentum of these pulses are evaluated exactly. The simple analytic structure of these pulses may prove useful in the detailed analysis of the above-mentioned experiments.

It is well-known that electromagnetic fields may be constructed from any suitable solution of the wave equation $(\nabla^2 - \partial_t^2)\psi = 0$ (we use the shorthand $\partial_t = c^{-1}\partial/\partial t$). For example, the vector potential may be taken to be

$$\mathbf{A} = \nabla \times [0, 0, \psi] = [\partial_y, -\partial_y, 0]\psi \tag{1}$$

and the scalar potential to be zero, in which case the electric and magnetic fields are

$$\begin{aligned} \mathbf{E} &= -\partial_t \mathbf{A} = [-\partial_y\partial_t, \partial_x\partial_t, 0]\psi \\ \mathbf{B} &= \nabla \times \mathbf{A} = [\partial_x\partial_z, \partial_y\partial_z, -\partial_x^2 - \partial_y^2]\psi. \end{aligned} \tag{2}$$

The electric field has only x and y components, so for a pulse with net propagation in the z-direction, these fields are transverse-electric (TE).

Ziolkowski [5] found a simple solution of the wave equation, namely

$$\psi_Z = \frac{ab}{\rho^2 + [a - i(z+ct)][b + i(z-ct)]}\psi_0 \tag{3}$$

and this wavefunction was used by Feng *et al* [6] to construct an electromagnetic pulse, using the vector potential $\mathbf{A} = \nabla \times [\psi, 0, 0]$. They evaluated the energy U of this pulse, but not its momentum.

When the net momentum P_z was evaluated later [7], it was found that $U > cP_z$. This (perhaps surprizing) result implied that the pulse could be Lorentz-transformed to a zero-momentum frame (not a *rest* frame). It was later shown that $U > cP_z$ for all three-dimensionally localized electromagnetic pulses [8]. The physical reason is the necessary convergence or spreading of localized solutions of the wave equation. The fact that $U > cP_z$ is in contradistinction to the Einstein photon [9] (for which energy is always c times the momentum, in any Lorentz frame) indicates that the Einstein photon cannot be a quantized version of a localized electromagnetic wavepacket.

We shall see that the TE pulse (and its dual, the TM pulse obtained by the duality transformation $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$) have zero angular momentum when $\psi = \psi_Z$. This same ψ can give nonzero angular momentum pulses, for example by taking the complex vector potential to be [7] $\mathbf{A} = \nabla \times [\pm i\psi, \psi, 0]$.

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However, in this letter we shall examine electromagnetic TE (or TM) pulses with $\mathbf{A}$ given by (1), in which ψ has explicit azimuthal dependence.

Reference [10] showed how solutions of the wave equation, with azimuthal dependence $e^{im\phi}$, may be obtained by a generalization of the Hillion [11] set of solutions. Those with $e^{im\phi}$ azimuthal dependence are

$$\psi = \left[\frac{\rho}{b+\mathrm{i}(z-ct)}\right]^{|m|} e^{im\phi}\frac{f(s)}{b+\mathrm{i}(z-ct)},\qquad s = \frac{\rho^2}{b+\mathrm{i}(z-ct)} - \mathrm{i}(z+ct). \tag{4}$$

(Note $m = 0$ and $f(s) = ab\psi_0/(s+a)$ gives ψ_Z.) In particular, we shall explore the properties of TE pulses based on

$$\psi_\pm = \frac{x \pm \mathrm{i}y}{b+\mathrm{i}(z-ct)}\psi_Z = \frac{\rho e^{\pm i\phi}}{b+\mathrm{i}(z-ct)}\psi_Z. \tag{5}$$

Consider first a TE pulse with vector potential given by (1) and the consequent fields given by (2), and suppose that ψ has $e^{im\phi}$ azimuthal dependence. Differentiations with respect to x and y then take the forms

$$\partial_x = \cos\phi\,\partial_\rho - \rho^{-1}\sin\phi\,\partial_\phi \to \cos\phi\,\partial_\rho - \mathrm{i}(m/\rho)\sin\phi,$$
$$\partial_y = \sin\phi\,\partial_\rho + \rho^{-1}\cos\phi\,\partial_\phi \to \sin\phi\,\partial_\rho + \mathrm{i}(m/\rho)\cos\phi. \tag{6}$$

From (2) and $E_\rho = E_x\cos\phi + E_y\sin\phi$, $E_\phi = -E_x\sin\phi + E_y\cos\phi$ we find that the complex electric field is

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho + \hat{\boldsymbol{\varphi}}E_\phi, \qquad E_\rho = -\frac{\mathrm{i}m}{\rho}\partial_t\psi, \qquad E_\phi = \partial_\rho\partial_t\psi. \tag{7}$$

The complex magnetic field has a longitudinal component as well:

$$\mathbf{B} = \hat{\boldsymbol{\rho}}B_\rho + \hat{\boldsymbol{\varphi}}B_\phi + \hat{\mathbf{z}}B_z, \qquad B_\rho = \partial_\rho\partial_z\psi,$$
$$B_\phi = \frac{\mathrm{i}m}{\rho}\partial_z\psi, \qquad B_z = \left(-\partial_\rho^2 - \rho^{-1}\partial_\rho + \frac{m^2}{\rho^2}\right)\psi. \tag{8}$$

(Physical fields are obtained by taking either real or imaginary parts of these expressions.)

The angular momentum density is

$$\mathbf{j} = \mathbf{r}\times\mathbf{p} = \frac{1}{4\pi c}\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) \tag{9}$$

where $\mathbf{p}$ is the momentum density. The total angular momentum $\mathbf{J}$ has its component along the direction of $\mathbf{P}$ (which is $[0, 0, P_z]$ in the cases considered here) invariant under Lorentz boosts and under change of origin [12]. Now

$$J_z = \int d^3r\, j_z = \int d^3r\, \rho p_\phi \tag{10}$$

where

$$4\pi c p_\phi = E_z B_\rho - E_\rho B_z. \tag{11}$$

When $m = 0$ we have $E_\rho = 0$ by (7), and E_z is always zero for TE pulses, so p_ϕ and j_z are zero when $m = 0$ (and in particular for TE pulses with $\psi = \psi_Z$, hence the assertion above that these pulses have zero angular momentum).

For the wavefunctions $\psi_\pm$ given in (5), we can evaluate the total energy and momentum of the pulse,

$$U = \frac{1}{8\pi}\int d^3r\,(E^2+B^2), \qquad cP_z = \frac{1}{4\pi}\int d^3r\,(\mathbf{E}\times\mathbf{B})_z \tag{12}$$

as well as the total angular momentum J_z given in (10). We find, using the techniques of [7], for either the real or imaginary parts of the complex fields in (7) and (8),

$$U = \frac{\pi}{16}\frac{3a+b}{b^2}\psi_0^2, \qquad cP_z = \frac{\pi}{16}\frac{3a-b}{b^2}\psi_0^2, \qquad cJ_z = \mp\frac{\pi}{8}\frac{a}{b}\psi_0^2. \tag{13}$$

The upper sign is for $m = +1$, the lower for $m = -1$.

Note that the sign of J_z is opposite to what may be expected: the $m = +1$ pulse wavefunction has azimuthal dependence $e^{i\phi}$, and the quantum-mechanical angular momentum operator $L_z = -\mathrm{i}\hbar(\mathbf{r}\times\nabla)_z = -\mathrm{i}\hbar\partial_\phi$ has $e^{i\phi}$ as an eigenstate, with eigenvalue $+\hbar$. Thus a particle with scalar wavefunction which has $e^{i\phi}$ azimuthal dependence will have positive angular momentum, whereas our electromagnetic pulse composed of vector fields derived from just such a wavefunction has negative angular momentum.

The helicity of the electromagnetic pulse depends on the sign of p_ϕ, which for our TE pulse is given by the sign of $-E_\rho B_z$. Consider wavefunctions of the form

$$\psi = e^{i\phi}\chi(\rho, z, t) \equiv e^{i\phi}(F + \mathrm{i}G). \tag{14}$$

This has real and imaginary parts

$$\mathrm{Re}(\psi) = \cos\phi F - \sin\phi G, \qquad \mathrm{Im}(\psi) = \sin\phi F + \cos\phi G. \tag{15}$$

Thus, from (7) and (8), taking the real parts as the physical fields,

$$E_\rho = \mathrm{Re}(-\mathrm{i}\rho^{-1}\partial_t\psi) = \rho^{-1}\,\mathrm{Im}(\partial_t\psi), \tag{16}$$

$$B_z = \mathrm{Re}\{(-\partial_\rho^2 - \rho^{-1}\partial_\rho + \rho^{-2})\psi\} = \mathrm{Re}\{(\partial_z^2 - \partial_t^2)\psi\} \tag{17}$$

(the last equality follows from the fact that ψ satisfies the wave equation). From (15), the average over ϕ of $-\rho E_\rho B_z$ is

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\,(-\rho E_\rho B_z) = \frac{1}{2}(\partial_t F)(\partial_z^2 - \partial_t^2)G - \tfrac{1}{2}(\partial_t G)(\partial_z^2 - \partial_t^2)F. \tag{18}$$

Since J_z is unchanged by Lorentz boosts along z [12], we may transform to the zero-momentum frame L_0 via

$$z = \gamma(z_0 + \beta ct_0), \qquad ct = \gamma(ct_0 + \beta z_0),$$
$$\gamma = (1-\beta^2)^{-\frac{1}{2}}, \qquad \beta = \frac{3a-b}{3a+b}. \tag{19}$$

(The transverse coordinates ρ and ϕ are unchanged by the Lorentz transformation.) Then

$$z - ct = \alpha^{-1}(z_0 - ct_0), \qquad z + ct = \alpha(z_0 + ct_0) \tag{20}$$

where

$$\alpha = \sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{\frac{3a}{b}}. \tag{21}$$

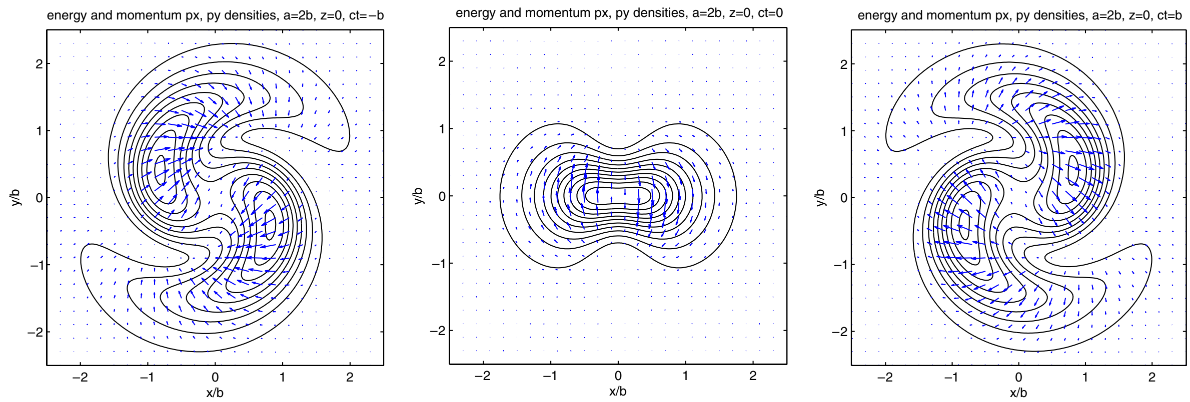

Figure 1. Energy density (contours) and the transverse p_x and p_y momentum densities (arrows) in the $z = 0$ plane, at three times $ct = -b$, 0 and b, corresponding to (a), (b) and (c). The parameters in these and all other figures are $m = +1$, $a = 2b$.

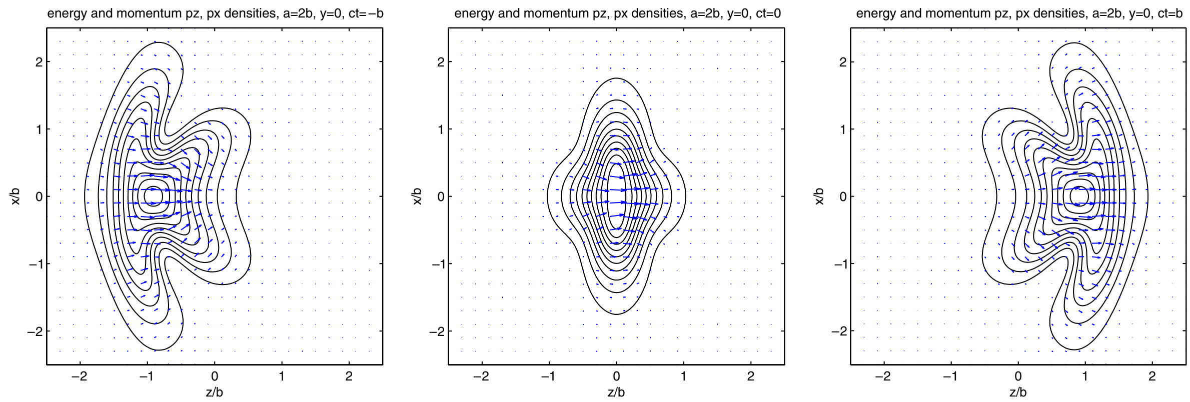

Figure 2. As for figure 1, but showing a longitudinal section (the $y = 0$ plane) and the longitudinal momentum component p_z, as well as one of the transverse momentum components (p_x).

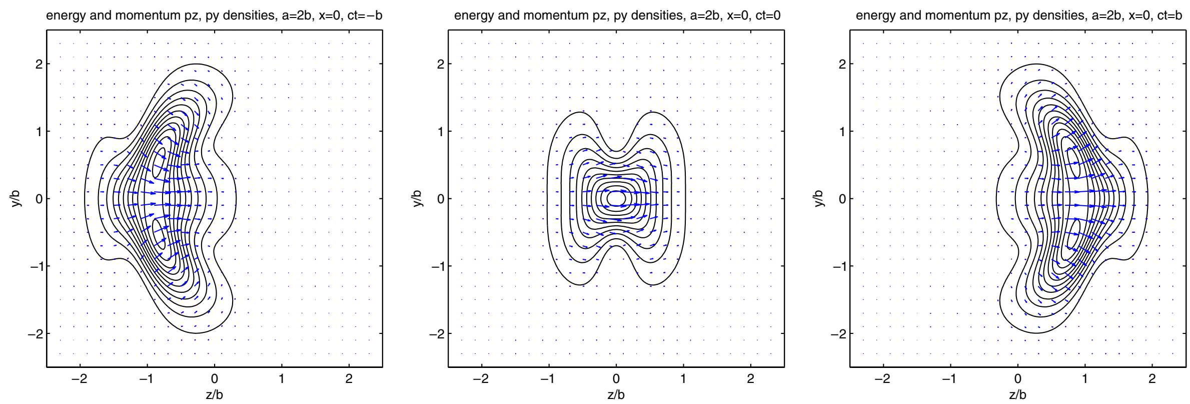

Figure 3. As for figure 2, now showing the p_y transverse momentum component in the $x = 0$ plane. Note that in this figure, as in figures 1 and 2, the arrows representing momentum density have been enlarged at $ct = \pm b$ by a factor of 6, for better visibility.

The wavefunction in this frame has the effective length parameters $a_0 = a/\alpha = (ab/3)^{\frac{1}{2}}$, $b_0 = b\alpha = (3ab)^{\frac{1}{2}}$:

$$\psi(\rho, \phi, z_0, t_0) = \frac{\rho \mathrm{e}^{\mathrm{i}\phi}}{b_0 + \mathrm{i}(z_0 - ct_0)} \times \frac{a_0 b_0 \alpha \psi_0}{\rho^2 + [a_0 - \mathrm{i}(z_0 + ct_0)][b_0 + \mathrm{i}(z_0 - ct_0)]}. \tag{22}$$

In L_0 the expression (18) takes a negative-definite form at time zero (J_z is independent of time [12]), and so the helicity is negative, in all frames.

We note in passing that (22) holds for an arbitrary Lorentz boost (i.e. any α), which can thus be viewed as equivalent to changing a to a/α, b to $b\alpha$ and ψ_0 to $\alpha\psi_0$. Making those replacements in the expressions in (13) leaves J_z unchanged, in accord with its Lorentz-invariance.

The helical nature of the pulses considered here is shown in energy density and momentum density plots (the latter also give the energy flux density, or Poynting vector, which is c^2 times the momentum density). Figures 1–3 show the energy density contours and the p_x, p_y and p_z momentum densities

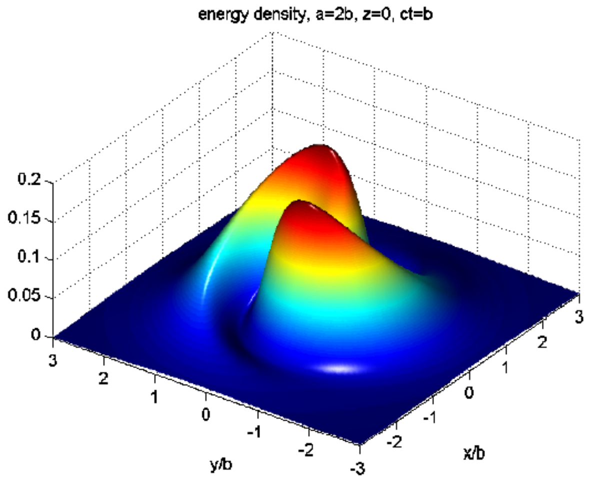

Figure 4. The energy density of the pulse, in the $z = 0$ (focal) plane at time $t = b/c$. Note the two arms of the energy density surface.

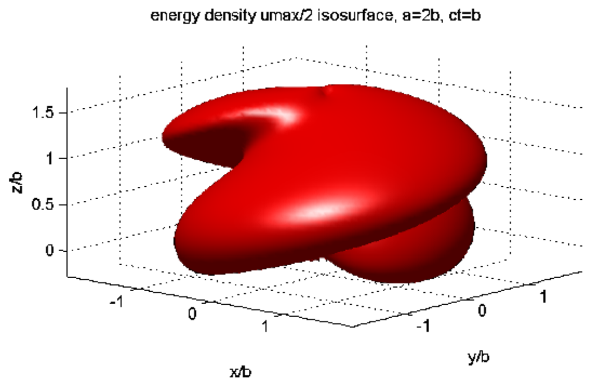

Figure 5. The energy density isosurface, $u = \frac{1}{2}u_{\mathrm{max}}$, at $t = b/c$. Note the two short right-handed screw threads which together make up the energy isosurface.

at the three times $ct = -b, 0, b$. Time zero corresponds to the pulse being centred on its focal region (at the origin). Figure 4 shows the distribution of energy in the $z = 0$ (focal) plane at time $t = b/c$, while figure 5 is an isosurface of the energy density at the same time. Figures 1(a), (c) and 5 together reveal the twists in the pulse: two short right-handed screw threads. Note that the figures are for the $m = +1$ pulse, which has negative J_z (which in the quantum particle case would imply a negative helicity), but the pulse twist is right-handed.

The helical structure of the pulses studied here correlates counterintuitively with their angular momentum. In the case of light beams, Allen *et al* [13] have shown that circularly polarized Laguerre–Gaussian beams with azimuthal dependence $e^{im\phi}$ have total angular momentum J_z proportional to $+\hbar m$ plus a spin component. An experimental demonstration of the effect of orbital and spin angular momentum was provided by O'Neil *et al* [14]. A recent calculation of (*inter alia*) the angular momentum of three types of generalized Bessel beams [15] also shows a component of J_z proportional to m. However, Molina-Terriza *et al* [16] have reported the transformation of a beam to one in which J_z and m have opposite signs.

There are other interesting aspects of the angular momentum of electromagnetic pulses:

(i) the wavefunction need not have an azimuthal dependence for the angular momentum to be nonzero; an example is provided by the pulses with vector potential $\mathbf{A} = \nabla \times [\pm\mathrm{i}, 1, 0]\psi_Z$ in [7], and
(ii) pulses based on a wavefunction with azimuthal dependence need not have a helical structure, even when their angular momentum is nonzero: for example the TE + iTM pulses based on $\psi_\pm$ have exactly twice the values of total energy, momentum and angular momentum given in (13), and yet their energy density u and momentum densities p_z, p_ρ and p_ϕ are all independent of ϕ [10].

References

[1] Mair A, Vaziri A, Weihs G and Zeilinger A 2001 Entanglement of the orbital angular momentum states of photons *Nature* **412** 313–6
[2] Molina-Terriza G, Torres J P and Torner L 2002 Management of the angular momentum of light: preparation of photons in multidimensional vector states of angular momentum *Phys. Rev. Lett.* **88** 0136011–4
[3] Leach J, Padgett M J, Barnett S M, Franke-Arnold S and Courtial J 2002 Measuring the orbital angular momentum of a single photon *Phys. Rev. Lett.* **88** 2579011–4
[4] Leach J, Courtial J, Skeldon K, Barnett S M, Franke-Arnold S and Padgett M J 2004 Interferometric methods to measure orbital and spin, or the total angular momentum of a single photon *Phys. Rev. Lett.* **92** 0136011–4
[5] Ziolkowski R W 1985 Exact solutions of the wave equation with complex source locations *J. Math. Phys.* **26** 861–3
[6] Feng S, Winful H G and Hellwarth R W 1999 Spatiotemporal evolution of focused single-cycle electromagnetic pulses *Phys. Rev.* E **59** 4630–9
[7] Lekner J 2003 Electromagnetic pulses which have a zero momentum frame *J. Opt. A: Pure Appl. Opt.* **5** L15–8
[8] Lekner J 2004 Energy and momentum of electromagnetic pulses *J. Opt. A: Pure Appl. Opt.* **6** 146–7
[9] Einstein A 1917/1972 On the quantum theory of radiation [English translation of *Phys. Zeits.* **18** 121, reprinted in F S Barnes (ed) *Laser Theory* (IEEE Press, New York)]
[10] Lekner J 2004 Localized electromagnetic pulses with azimuthal dependence *J. Opt. A: Pure Appl. Opt.* **6** 711–6
[11] Hillion P 1993 Generalized phases and nondispersive waves *Acta Appl. Math.* **30** 35–45
[12] Lekner J 2004 Angular momentum of electromagnetic pulses *J. Opt. A: Pure Appl. Opt.* **6** S128–33
[13] Allen L, Beijersbergen M W, Spreeuw R J C and Woerdman J P 1992 Orbital angular momentum of light and the transformation of Laguerre–Gaussian modes *Phys. Rev.* A **45** 8185–9
[14] O'Neil A T, MacVicar I, Allen L and Padgett M J 2002 Intrinsic and extrinsic nature of the orbital angular momentum of a light beam *Phys. Rev. Lett.* **88** 0536011–4
[15] Lekner J 2004 Invariants of three types of generalized Bessel beams *Pure Appl. Opt.* **6** 837–43
[16] Molina-Terriza G, Recolons J, Torres J P, Torner L and Wright E W 2001 Observation of the dynamical inversion of the topological charge of an optical vortex *Phys. Rev. Lett.* **87** 0239021–4