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Energy and momentum of electromagnetic pulses

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Abstract
We show that free-space pulse solutions of Maxwell's equations which are localized in space and time have energy greater than c times their momentum. Thus a Lorentz transformation to a zero-momentum frame is always possible, in contradistinction to Einstein's light quantum, for which a zero-momentum frame does not exist. However, free-space pulse solutions of Maxwell's equations are not subluminal: their momentum is less than their energy divided by c due to necessary spreading or convergence.

Keywords: electromagnetic pulses, Lorentz transformations, light quanta

In a recent letter [1] the author has shown that certain free-space exact pulse solutions of Maxwell's equations have momentum smaller than their energy divided by c. For such pulses a Lorentz transformation to a zero-momentum frame is possible. It was conjectured in that letter that the result holds for all localized pulses of finite energy. In this paper we show that the conjecture is correct. This shows that there can be no semiclassical correspondence between electromagnetic pulses and the Einstein light quantum [2], for which energy equals c times momentum, in all inertial frames of reference.

The results of [1] were all based on a certain set of solutions of the wave equation [3]. There exist broader classes of exact solutions [4–9]. Here we shall give proofs for three types of pulses which are valid for any such solutions, and then give a general proof valid for any free-space pulse of finite energy.

The results of this paper are restricted to *free-space* propagation of electromagnetic pulses. In two-conductor transmission lines (with vacuum as dielectric), dispersionless propagation of TEM pulses is possible, and these pulses have energy equal to momentum times c, provided resistive losses in the conductors can be neglected (see, for example, section 8.2 and problems 8.1 and 8.2 of [10]). The purely transverse and dispersion-free nature of these pulses is related to the one-dimensional propagation of energy and momentum, which is exclusively along the transmission line.

We begin with the TE + iTM pulse, discussed in detail in [1]. This has vector potential $\boldsymbol{A} = \nabla \times [0, 0, \psi]$, and zero scalar potential. The electromagnetic fields are $\boldsymbol{B} = \nabla \times \boldsymbol{A} + \mathrm{i}\partial_t \boldsymbol{A}$ (∂_t stands for $\partial/\partial(ct)$) and $\boldsymbol{E} = \mathrm{i}\boldsymbol{B}$. For such solutions the energy and momentum densities (in Gaussian units) are [1]

$$u = \frac{1}{8\pi}|\boldsymbol{E}|^2 = \frac{1}{8\pi}|\boldsymbol{B}|^2, \qquad \boldsymbol{p} = \frac{\mathrm{i}}{8\pi c}\boldsymbol{E}\times\boldsymbol{E}^* = \frac{\mathrm{i}}{8\pi c}\boldsymbol{B}\times\boldsymbol{B}^*. \tag{1}$$

(Actual physical solutions are the real or imaginary parts of $\boldsymbol{E}$ and $\boldsymbol{B}$: when $\boldsymbol{E} = \boldsymbol{E}_\mathrm{r} + \mathrm{i}\boldsymbol{E}_\mathrm{i} = \mathrm{i}(\boldsymbol{B}_\mathrm{r} + \mathrm{i}\boldsymbol{B}_\mathrm{i})$ we have $E_\mathrm{r}^2 + B_\mathrm{r}^2 = E_\mathrm{r}^2 + E_\mathrm{i}^2 = |\boldsymbol{E}|^2 = |\boldsymbol{B}|^2 = E_\mathrm{i}^2 + B_\mathrm{i}^2$ and $\boldsymbol{E}_\mathrm{r} \times \boldsymbol{B}_\mathrm{r} = \frac{\mathrm{i}}{2}\boldsymbol{E} \times \boldsymbol{E}^* = \frac{\mathrm{i}}{2}\boldsymbol{B} \times \boldsymbol{B}^* = \boldsymbol{E}_\mathrm{i} \times \boldsymbol{B}_\mathrm{i}$.) In [1] the energy density and momentum density are given for the case where ψ is independent of the azimuthal angle ϕ:

$$u = \frac{1}{8\pi}\{|\partial_\rho\partial_z\psi|^2 + |\partial_\rho\partial_t\psi|^2 + |\partial_z^2\psi - \partial_t^2\psi|^2\} \tag{2}$$

$$p_z = -\frac{1}{4\pi c}\,\mathrm{Re}\{(\partial_\rho\partial_t\psi^*)(\partial_\rho\partial_z\psi)\}. \tag{3}$$

The x and y components of the momentum density integrate to zero, so the total momentum is $[0, 0, P_z]$. We see from (2) and (3) that

$$8\pi(u - cp_z) = |\partial_\rho(\partial_z + \partial_t)\psi|^2 + |(\partial_z - \partial_t)(\partial_z + \partial_t)\psi|^2 \geqslant 0. \tag{4}$$

For general ψ, ∂_ρ is to be replaced by $\partial_x + \mathrm{i}\partial_y$. There are two ways in which the total energy $U = \int \mathrm{d}^3 r u$ can equal c times the non-zero component of the total momentum, $P_z = \int \mathrm{d}^3 r p_z$: either $(\partial_z + \partial_t)\psi = 0$ everywhere and at all times, or $\{(\partial_x + \mathrm{i}\partial_y)\psi = 0$ and $(\partial_z - \partial_t)\psi = 0\}$ everywhere and at all times. The first condition is satisfied by $\psi = f(x, y, z - ct)$. For such a functional form the wave equation implies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \tag{5}$$

That is, f is harmonic in x and y, and thus cannot be localized in the transverse directions. The other alternative leads to the form $\psi = f(x+iy, z+ct)$, again not localized in the transverse directions, because f satisfies (5). Thus U is greater than cP_z for three-dimensionally localized TE + iTM pulses based on any solution of the wave equation.

The positive angular momentum pulse considered in [1] also has energy and momentum given by (1): it has $\boldsymbol{A} = \nabla \times [i\psi, \psi, 0]$, $\boldsymbol{B} = \nabla \times \boldsymbol{A} + i\partial_t \boldsymbol{A}$, and $\boldsymbol{E} = i\boldsymbol{B}$. For pulses of this type we find, after using the fact that ψ must obey the wave equation,

$$8\pi(u - cp_z) = |(\partial_z + \partial_t)^2\psi|^2 + |(\partial_x - i\partial_y)(\partial_z + \partial_t)\psi|^2 \geqslant 0. \tag{6}$$

We obtain equality of u with cp_z when $(\partial_z + \partial_t)\psi = 0$. This condition requires ψ to have the functional form $f(x, y, z - ct)$. Since $\nabla^2\psi = \partial_t^2\psi$, this f satisfies (5), and so cannot be localized transversely.

As a final example from [1], we take $\boldsymbol{A} = \nabla \times [\psi, 0, 0]$, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{E} = -\partial_t \boldsymbol{A}$ and real ψ. Feng *et al* [11] evaluated the energy (but not the momentum) of this pulse when ψ is taken as the real or imaginary part of the complex Ziolkowski solution of the wave equation [3], namely

$$\psi_Z = \{\rho^2 + [a - i(z + ct)][b + i(z - ct)]\}^{-1}. \tag{7}$$

Here we find (for any real ψ)

$$8\pi(u - cp_z) = [(\partial_y^2 + \partial_z^2 + \partial_z\partial_t)\psi]^2 + (\partial_x\partial_y\psi)^2 + (\partial_x\partial_z\psi)^2 + (\partial_y\partial_t\psi)^2 \geqslant 0. \tag{8}$$

Equality of u and cp_z is possible only for ψ independent of x and y, and satisfying $(\partial_z + \partial_t)\psi = 0$, i.e. of the plane wave form $g(z - ct)$.

The above examples demonstrate the truth of the conjecture (that energy is always greater than c times momentum for pulses localized in three dimensions) for three classes of pulse. To obtain a general proof we shall consider the energy and momentum densities in terms of the field components, instead of in terms of a solution of the wave equation, as was done above. The densities are $u = (E^2 + B^2)/8\pi$ and $\boldsymbol{p} = \boldsymbol{E} \times \boldsymbol{B}/4\pi c$, with real fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. If the z-axis is taken along the direction of the total momentum vector $\boldsymbol{P}$ (which of course is a constant [1]), we have

$$\begin{aligned} 8\pi(u - cp_z) &= \boldsymbol{E}^2 + \boldsymbol{B}^2 - 2(\boldsymbol{E} \times \boldsymbol{B})_z \\ &= E_x^2 + E_y^2 + E_z^2 + B_x^2 + B_y^2 + B_z^2 - 2(E_xB_y - E_yB_x) \\ &= (E_x - B_y)^2 + (E_y + B_x)^2 + E_z^2 + B_z^2 \geqslant 0. \end{aligned} \tag{9}$$

Thus $U = cP_z$ requires $E_z = 0 = B_z$ (i.e. purely transverse fields, everywhere) and also $\{E_x = B_y, E_y = -B_x\}$, everywhere. Maxwell's equations then imply that $(\partial_z + \partial_t)E_x = 0$ and $(\partial_z + \partial_t)E_y = 0$, i.e. both must be of the form $f(x, y, z-ct)$, and thus also that equation (5) is to hold for both E_x and E_y. Therefore neither of the two independent transverse components E_x and E_y can be localized transversely, since both are harmonic in x and y.

We have thus shown that free-space electromagnetic pulses which (at a given time) are localized in three-dimensions necessarily have energy greater than c times their momentum. Since $\boldsymbol{P}$ and U/c form a four-vector, a Lorentz boost at speed c^2P_z/U will transform a given pulse to its zero-momentum frame, in contrast to the Einstein light quanta, for which a zero-momentum frame does not exist.

The theorem just proved does not imply subluminal propagation: the propagation is at speed c, but the necessary spreading (or convergence) of the localized pulse makes the net momentum less than the energy divided by c.

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