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LETTER TO THE EDITOR

Electromagnetic pulses which have a zero momentum frame

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Abstract
One set of the Ziolkowski family of exact solutions of the wave equation is shown to represent pulses propagating with momentum smaller than energy$/c$. This is explicitly demonstrated for special cases by calculating the total electromagnetic momentum and energy. Since the ratio of momentum to energy is a constant smaller than c^{-1}, there exists a Lorentz transformation to a frame in which the total momentum is zero. In the zero-momentum frame the fields are those of an annular pulse converging onto or diverging from a focal region.

Keywords: Electromagnetic pulses, photons, Lorentz transformations

It is axiomatic in special relativity that there is no rest frame for light: the speed of light is the same in every inertial frame. It is known, however, that electromagnetic energy can travel at less than the speed of light [1], and here we shall show that there exists a Lorentz frame L_0 in which the total pulse electromagnetic momentum is zero, for a class of solutions of the Maxwell equations.

Ziolkowski [2] obtained exact solutions of the wave equation $\nabla^2\psi = c^{-2}\partial^2\psi/\partial t^2$ in the form ($\rho = \sqrt{x^2+y^2}$ is the distance from the propagation axis)

$$\psi(\boldsymbol{r},t) = \int_0^\infty \mathrm{d}k\, F(k) \times \frac{\exp\{\mathrm{i}k(z+ct) - k\rho^2/[b+\mathrm{i}(z-ct)]\}}{b+\mathrm{i}(z-ct)}. \tag{1}$$

With $F(k) = ab\mathrm{e}^{-ka}$ one obtains the particularly simple solution [2–4]

$$\psi(\boldsymbol{r},t) = \frac{ab}{\rho^2 + [a-\mathrm{i}(z+ct)][b+\mathrm{i}(z-ct)]}\psi_0 \tag{2}$$

where ψ_0 is the wavefunction value at the space–time origin. Feng *et al* [4] have called the electromagnetic fields derived from (2) 'focused single-cycle electromagnetic pulses'. We shall show in the next paragraph that, for an arbitrary electromagnetic pulse, the total momentum $\boldsymbol{P}$ and total energy U are both constant in time. We then show that the fields derived from (2) have $cP_z < U$. Thus a Lorentz transformation to a zero-momentum frame L_0 is possible, and in that frame the fields represent an annular pulse, converging for $t_0 < 0$, diverging for $t_0 > 0$. (In the 'laboratory' frame the wavefunction (2) gives fields in which there is net forward or backward propagation, in general.)

It follows from Maxwell's equations that the total energy $U = \int \mathrm{d}^3r\, u(\boldsymbol{r},t)$ is independent of time (∂_t denotes differentiation with respect to ct): $4\pi\partial_t U = \int \mathrm{d}^3r\,(\boldsymbol{E}\cdot\nabla\times\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\times\boldsymbol{E}) = -\int \mathrm{d}^3r\,\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})$ (real fields) from the Maxwell curl equations, and this integral can be expressed as a surface integral at infinity, which is zero at finite times. Likewise, again with real fields and $\boldsymbol{P} = \int \mathrm{d}^3r\,\boldsymbol{p}(\boldsymbol{r},t)$, $4\pi c\partial_t\boldsymbol{P} = -\int \mathrm{d}^3r\,[\boldsymbol{E}\times(\nabla\times\boldsymbol{E}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{B})]$, and integrations by parts show that this is also zero at finite times. Thus the total energy and total momentum are constant in time, as we would expect. If the ratio cP_z/U is less than unity, as we shall demonstrate it is in particular cases below, we can Lorentz-transform to the zero-momentum frame.

Given solutions of the wave equation, solutions of Maxwell's equations can be obtained as $\boldsymbol{E} = -\nabla\Phi - \partial_t\boldsymbol{A}$, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, where Φ and all components of $\boldsymbol{A}$ satisfy $\nabla^2\psi = \partial_t^2\psi$, provided the Lorentz condition $\nabla\cdot\boldsymbol{A} + \partial_t\Phi = 0$ holds [5]. For example, we can take $\Phi = 0$, $\boldsymbol{A} = \nabla\times[0,0,\psi] = [\partial_y, -\partial_x, 0]\psi$; this gives a TE field $\boldsymbol{E} = [-\partial_y\partial_t, \partial_x\partial_t, 0]\psi$, $\boldsymbol{B} = [\partial_x\partial_z, \partial_y\partial_z, -\partial_x^2 - \partial_y^2]\psi$. A TM field is obtained by the duality transformation $\boldsymbol{E}\to\boldsymbol{B}$, $\boldsymbol{B}\to-\boldsymbol{E}$: $\boldsymbol{E} = \nabla\times\boldsymbol{A}$, $\boldsymbol{B} = \partial_t\boldsymbol{A}$. The combination

TE + iTM has $\boldsymbol{E} = -\partial_t \boldsymbol{A} + \mathrm{i}\nabla\times\boldsymbol{A}$, $\boldsymbol{B} = \nabla\times\boldsymbol{A} + \mathrm{i}\partial_t\boldsymbol{A}$, i.e. $\boldsymbol{E} = \mathrm{i}\boldsymbol{B}$ (TE − iTM gives $\boldsymbol{E} = -\mathrm{i}\boldsymbol{B}$). In the monochromatic beam case these combinations give *steady beams*, in which the electromagnetic energy density u and momentum density $\boldsymbol{p}$ (=Poynting vector/c^2) do not oscillate in time [1, 6]. As in the steady beam case, the $\boldsymbol{E} = \pm\mathrm{i}\boldsymbol{B}$ solutions have (taking either $\mathrm{Re}(\boldsymbol{E},\boldsymbol{B})$ or $\mathrm{Im}(\boldsymbol{E},\boldsymbol{B})$ as the physical fields)

$$\begin{aligned} u &= \frac{1}{8\pi}|\boldsymbol{E}|^2 = \frac{1}{8\pi}|\boldsymbol{B}|^2, \\ \boldsymbol{p} &= \frac{\mathrm{i}}{8\pi c}\boldsymbol{E}\times\boldsymbol{E}^* = \frac{\mathrm{i}}{8\pi c}\boldsymbol{B}\times\boldsymbol{B}^*. \end{aligned} \tag{3}$$

When $\psi(\boldsymbol{r},t)$ is independent of the azimuthal angle ϕ, and $\boldsymbol{A} = [\partial_y, -\partial_x, 0]\psi$, we find

$$\begin{aligned} u &= \frac{1}{8\pi}\{|\partial_\rho\partial_z\psi|^2 + |\partial_\rho\partial_t\psi|^2 + |\partial_z^2\psi - \partial_t^2\psi|^2\}, \\ p_z &= -\frac{1}{4\pi c}\,\mathrm{Re}\{(\partial_\rho\partial_t\psi^*)(\partial_\rho\partial_z\psi)\} \end{aligned} \tag{4}$$

and $p_x = p_\rho\cos\phi - p_\phi\sin\phi$, $p_y = p_\rho\sin\phi + p_\phi\cos\phi$, where the radial and azimuthal components of the momentum density are given by

$$\begin{aligned} p_\rho &= \frac{1}{4\pi c}\,\mathrm{Re}\{(\partial_\rho\partial_t\psi^*)(\partial_z^2\psi - \partial_t^2\psi)\}, \\ p_\phi &= \frac{1}{4\pi c}\,\mathrm{Im}\{(\partial_\rho\partial_z\psi^*)(\partial_z^2\psi - \partial_t^2\psi)\}. \end{aligned} \tag{5}$$

Figures 1(a) and (b) show contours of u and field plots of p_z, p_ρ for $a = b$ and $a = 2b$ at $ct = 0$ and $3b$.

Since U and $\boldsymbol{P}$ are independent of time, we can evaluate them at $t = 0$. For ψ given by (2) and $A = [\partial_y\psi, -\partial_x\psi, 0]$ we have from (4) that, for the TE + iTM pulse,

$$u(\boldsymbol{r},0) = \frac{(ab\psi_0)^2}{\pi} \times \frac{r^4 + 2(a^2+b^2-ab)\rho^2 + (a^2+b^2)z^2 + (ab)^2}{[r^4 + 2ab\rho^2 + (a^2+b^2)z^2 + (ab)^2]^3} \tag{6}$$

$$cp_z(\boldsymbol{r},0) = \frac{(ab\psi_0)^2}{\pi}\frac{(a^2-b^2)\rho^2}{[r^4 + 2ab\rho^2 + (a^2+b^2)z^2 + (ab)^2]^3} \tag{7}$$

where $r^2 = \rho^2 + z^2$. The integrations in spherical polar coordinates (r,θ,ϕ) are helped by the substitution $\cos\theta = \frac{r^2+ab}{(a-b)r}\tan\chi$ ($a > b > 0$ is assumed). We find

$$U = \frac{\pi}{8}\frac{a+b}{ab}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-b}{ab}\psi_0^2. \tag{8}$$

(The transverse components of momentum integrate to zero.) Thus $cP_z/U = (a-b)/(a+b)$ is less than unity: the net momentum of the electromagnetic field is less than its energy/c. We can interpret c^2P_z/U as an average energy velocity [1, 5] βc, $\beta = (a-b)/(a+b)$. Independently of this interpretation, the fact that $\beta < 1$ implies that we can transform to the Lorentz frame L_0 in which the total momentum is zero. In this frame we have ρ unchanged, and

$$z = (z_0 + \beta ct_0)\big/\sqrt{1-\beta^2} \qquad ct = (ct_0 + \beta z_0)\big/\sqrt{1-\beta^2} \tag{9}$$

and so, replacing $(1+\beta)/(1-\beta)$ by a/b,

$$z + ct = \sqrt{\frac{a}{b}}(z_0 + ct_0), \qquad z - ct = \sqrt{\frac{b}{a}}(z_0 - ct_0). \tag{10}$$

The wavefunction in (2) thus becomes, in the zero-momentum frame,

$$\psi(\boldsymbol{r}_0,t_0) = \frac{ab\psi_0}{\rho^2 + \big[\sqrt{ab} - \mathrm{i}(z_0+ct_0)\big]\big[\sqrt{ab} + \mathrm{i}(z_0-ct_0)\big]} \tag{11}$$

which gives equal weight to the forward and backward propagations in the scalar wave.

Feng *et al* [4] have taken vector potential $\boldsymbol{A} = \nabla\times[\psi,0,0]$, and fields $\boldsymbol{E} = -\partial_t\boldsymbol{A}$, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. For this pulse the calculations are more complicated than for the TE + iTM pulse above. We find, for both the real and imaginary parts of ψ,

$$U = \frac{\pi}{64}\frac{(a+b)(3a^2-2ab+3b^2)}{(ab)^2}\psi_0^2 \tag{12}$$

$$cP_z = \frac{\pi}{64}\frac{(a-b)(3a^2+2ab+3b^2)}{(ab)^2}\psi_0^2. \tag{13}$$

(The expression for U is in agreement with equation (3.6) of [4].) The Lorentz boost to L_0 is

$$\beta = \frac{a-b}{a+b}\frac{3a^2+2ab+3b^2}{3a^2-2ab+3b^2}. \tag{14}$$

The wavefunction in the zero-momentum frame is now

$$\psi(\boldsymbol{r}_0,t_0) = \frac{ab\psi_0}{\rho^2 + [a/\alpha - \mathrm{i}(z_0+ct_0)][b\alpha + \mathrm{i}(z_0-ct_0)]} \tag{15}$$

where

$$\alpha = \sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{\frac{a(3a^2+b^2)}{b(a^2+3b^2)}}. \tag{16}$$

The angular momentum density is $\boldsymbol{j} = \boldsymbol{r}\times\boldsymbol{p}$ [5, 7] where $\boldsymbol{p}$ is the momentum density; the angular momentum of a pulse is $\boldsymbol{J} = \int \mathrm{d}^3r\,\boldsymbol{j}$. For the TE + iTM and TE classical electromagnetic pulses described above, all the components of $\boldsymbol{J}$ are zero. By analogy with a 'steady' beam which in the plane-wave limit is circularly polarized everywhere [8], we construct the $\boldsymbol{E} = \mathrm{i}\boldsymbol{B}$ pulse

$$\begin{gathered} \boldsymbol{A} = \nabla\times[-\mathrm{i}\psi,\psi,0], \qquad \boldsymbol{E} = -\partial_t\boldsymbol{A} + \mathrm{i}\nabla\times\boldsymbol{A}, \\ \boldsymbol{B} = \nabla\times\boldsymbol{A} + \mathrm{i}\partial_t\boldsymbol{A}. \end{gathered} \tag{17}$$

The energy density and momentum density are again given by (3). The complex magnetic field is

$$\begin{aligned} \boldsymbol{B} = [&(\partial_x + \mathrm{i}\partial_y)\partial_y + \mathrm{i}(\partial_z - \partial_t)\partial_z, -(\partial_x + \mathrm{i}\partial_y)\partial_x - (\partial_z - \partial_t)\partial_z, \\ &-\mathrm{i}(\partial_x + \mathrm{i}\partial_y)(\partial_z - \partial_t)]\psi \end{aligned} \tag{18}$$

and when ψ is given by (2) we find the energy and z-components of momentum and angular momentum of the pulse to be

$$\begin{gathered} U = \frac{\pi}{8}\frac{3a+b}{b^2}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{3a-b}{b^2}\psi_0^2, \\ cJ_z = -\frac{\pi}{4}\frac{a}{b}\psi_0^2. \end{gathered} \tag{19}$$

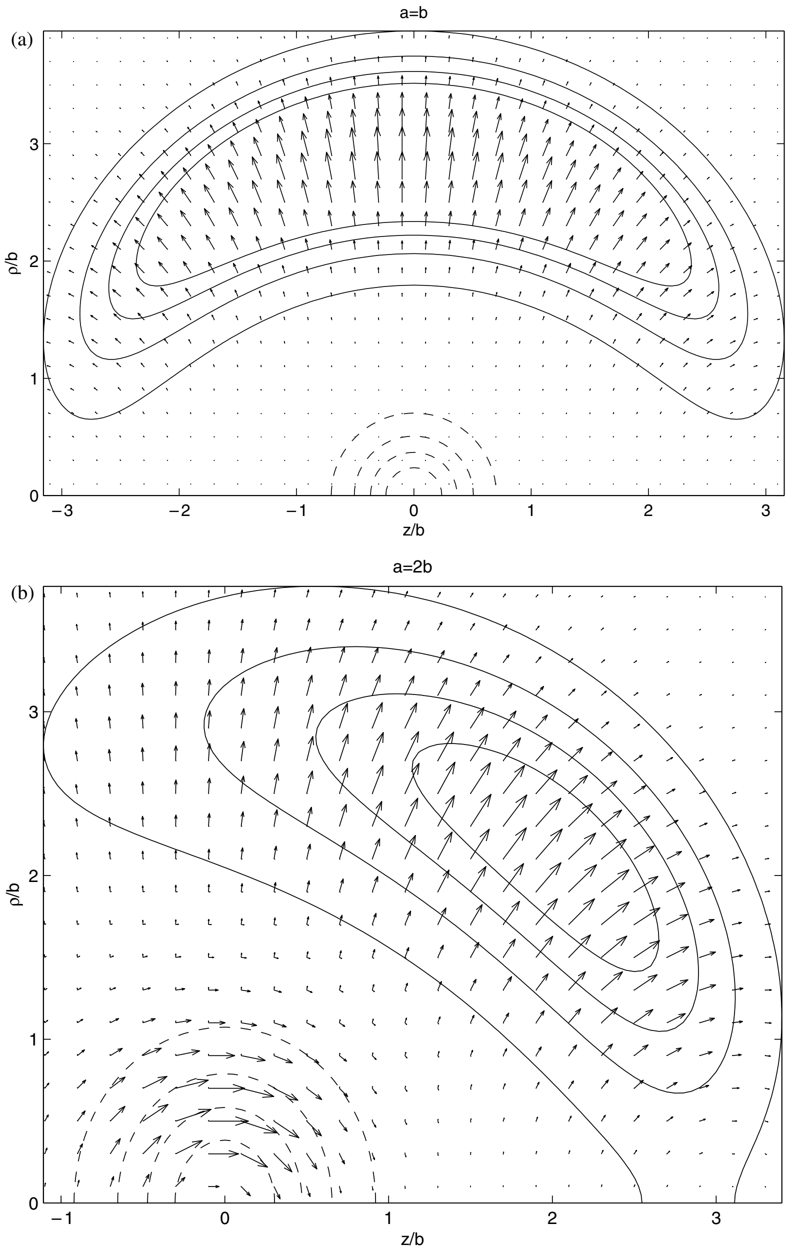

Figure 1. Both figures show contours of energy density for the TE + iTM pulse with $a = b$ (a) and $a = 2b$ (b). The contours are at 0.8, 0.6, 0.4 and 0.2 of the maximum at the given value of ct. In each case the $t = 0$ contours are dashed curves, and the $ct = 3b$ contours are solid curves. The three-dimensional contour surfaces are obtained by rotating the figures about the horizontal (z) axis. In the $a = b$ case the pulse is diverging (for $t > 0$) symmetrically from the origin, and the pulse momentum is zero. In the $a = 2b$ case there is a net momentum in the z direction, and the pulse energy density is asymptotically maximum on a cone of half-angle $\theta_m \approx 46.7°$. The general expression for the asymptotic angle at which the energy density is maximum is $\sin^2(\theta_m/2) = [5a - 3b - \sqrt{25a^2 - 46ab + 25b^2}]/8(a - b)$. The arrows indicate the magnitude and direction of the projection of the momentum density onto a plane which includes the z-axis (i.e. the azimuthal component is not shown). The momentum density is identically zero at $t = 0$ when $a = b$. At $ct = 3b$ in the $a = 2b$ case, the magnitude of the momentum density has been increased by a factor of 22 (relative to the $t = 0$ values) for better visibility.

The sign of J_z is reversed if one takes the vector potential and fields to be

$$\begin{aligned} \boldsymbol{A} &= \nabla \times [\mathrm{i}\psi, \psi, 0], \\ \boldsymbol{E} &= -\partial_t \boldsymbol{A} + \mathrm{i}\nabla \times \boldsymbol{A}, \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} + \mathrm{i}\partial_t \boldsymbol{A}. \end{aligned} \tag{20}$$

The results are then as in (19) with a and b interchanged, and the sign of J_z changed:

$$\begin{aligned} U &= \frac{\pi}{8}\frac{a+3b}{a^2}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-3b}{a^2}\psi_0^2, \\ cJ_z &= \frac{\pi}{4}\frac{b}{a}\psi_0^2. \end{aligned} \tag{21}$$

In general, since P_z and U have been shown to be independent of time, their ratio is also independent of time. Whenever $cP_z < U$, we can transform to a zero-momentum frame. We suspect that this is possible for all solutions which represent pulses converging onto and then diverging from a focal region (for example, those of Kiselev and Perel [9]).

According to Adlard *et al* [10], 'given any classical solution of the source-free Maxwell's equations it is possible to write down a corresponding quantum mechanical one-photon state'. They use a particular case of the Ziolkowski solutions to show that 'single-photon states with arbitrarily high powers of asymptotic falloff can be explicitly constructed'. The quantum fields resulting from the Ziolkowski family of solutions may thus represent (for $t > 0$) photons diverging from a focal region. They are rather different from the textbook photon, which is monochromatic and unidirectional, with $U = \hbar\omega = cP_z$ [11], and $J_z = \pm\hbar$. Although there are no sustained oscillations in the pulses derived from (2), we can associate an effective frequency with the pulse as follows: since $(c\boldsymbol{P}, U)$ is a four-vector, the pulse energy in the zero-momentum frame L_0 is $U_0 = (U - \beta c P_z)/\sqrt{1-\beta^2}$. From (21), for example, we obtain (with $\beta = (a-3b)/(a+3b)$)

$$U_0 = \frac{\pi}{4}\frac{b}{a}\sqrt{\frac{3}{ab}}\psi_0^2. \tag{22}$$

Thus, for the vector potential and fields in (20),

$$U = \frac{a+3b}{2\sqrt{3ab}}U_0, \qquad cP_z = \frac{a-3b}{2\sqrt{3ab}}U_0, \qquad cJ_z = \sqrt{\frac{ab}{3}}U_0. \tag{23}$$

If we set $J_z = \hbar$ and $U_0 = \hbar\omega_0$, the resulting angular frequency in L_0 is

$$\omega_0 = U_0/J_z = c\sqrt{\frac{3}{ab}}. \tag{24}$$

The effective angular frequency in the laboratory frame is

$$\omega = U/J_z = c\frac{a+3b}{2ab}. \tag{25}$$

The frequency ratio ω/ω_0 is not given by the usual Doppler expression $\sqrt{\frac{1+\beta}{1-\beta}}$, since that applies only to monochromatic plane waves.

We have shown that electromagnetic pulses exist which have $cP_z < U$, and which can therefore be transformed to a zero-momentum frame. We suspect that this will hold for all localized pulses of finite energy, since these must converge or diverge to some extent. If the same is true for photons, the Einstein picture of light quanta [11] will need modification. We intend to explore the implications elsewhere.

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References

[1] Lekner J 2002 *J. Opt. A: Pure Appl. Opt.* **4** 491–9
[2] Ziolkowski R W 1985 *J. Math. Phys.* **26** 861–3
Ziolkowski R W 1989 *Phys. Rev.* A **39** 2005–33
[3] Hellwarth R W and Nouchi P 1996 *Phys. Rev.* E **54** 889–95
[4] Feng S, Winful H G and Hellwarth R W 1999 *Phys. Rev.* E **59** 4630–49
[5] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
[6] Lekner J 2001 *J. Opt. A: Pure Appl. Opt.* **3** 407–12
Lekner J 2002 *J. Opt. A: Pure Appl. Opt.* **4** 219–20
[7] Allen L, Padgett M J and Babiker M 1999 *Prog. Opt.* **39** 291–372
[8] Lekner J 2003 *J. Opt. A: Pure Appl. Opt.* **5** 6–14
[9] Kiselev A P and Perel M V 2000 *J. Math. Phys.* **41** 1934–55
[10] Adlard C, Pike E R and Sarkar S 1997 *Phys. Rev. Lett.* **79** 1585–7
[11] Einstein A 1917 *Phys. Z.* **18** 121–8 (Engl. transl. 1972 *Laser Theory* ed F S Barnes (New York: IEEE) (reprinted))