An upper bound on acoustic reflectivity, and the Rayleigh approximation

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(Received 23 December 1988; accepted for publication 3 July 1989)

Reflection of plane acoustic compressional waves at a stratified transitional layer between two fluid media is treated by means of a nonlinear differential equation for the reflection amplitude. When the normal component of the wave vector divided by the local density changes monotonically, the reflectance is shown to be no greater than that at a sharp transition between the same two media (at the same angle of incidence). A related Riccati-type differential equation for the reflection amplitude leads to the Rayleigh (or weak-reflection) approximation. This approximation is simple, easy to evaluate, and works well at all wavelengths provided that the reflection is weak.

PACS numbers: 43.20.Fn, 43.20.Bi, 43.30.Bp

INTRODUCTION

It is known that particle waves (satisfying the Schrödinger equation) reflect less from a gradual transition between two media than from a sharp transition. This intuitively plausible result is also generally true for the electromagnetic *s* wave but holds only under restricted conditions for the electromagnetic *p* wave (Ref. 1, Sec. 5-4). For acoustic waves the situation is more complex still, as will be shown here. At normal incidence the reflection from a transitional layer will be less than that from an abrupt transition between the same two bounding media, provided the product of the density ρ and the local phase velocity c is a monotonically increasing or decreasing function of the depth z. At a general angle of incidence, we will show that, if ρc and c both increase or both decrease monotonically, the reflectivity will be no greater than that from an abrupt transition.

The above results are derived from a Riccati-type equation satisfied by a quantity related to the reflection amplitude; this equation was introduced by Kofink[2] and used by Brekhovskikh[3] in the calculation of wave reflection (for references to earlier work in related fields see Chap. 5 of Ref. 1). A similar approach was used in Ref. 1 to rederive the Rayleigh approximation[4] for electromagnetic waves of both polarizations. This same approach is adapted here to the reflection of acoustic waves in Sec. IV, and compared with a simple solvable model in Sec. V.

I. PROPAGATION AND REFLECTION IN STRATIFIED MEDIA

Consider sound propagation in an inhomogeneous medium. Let $\rho + \rho_a, p + p_a$ be the density and pressure within the medium, with ρ and p being the equilibrium values, and ρ_a and p_a the time-dependent oscillating variations associated with the acoustic wave. The linearized wave equation for p_a is[5]

$$\nabla^2 p_a - \frac{1}{c^2}\frac{\partial^2 p_a}{\partial t^2} - \frac{1}{\rho}\nabla\rho_a\cdot\nabla p_a = 0, \tag{1}$$

where the adiabatic derivative $(\partial p/\partial\rho)_s = c^2$ gives the square of the local value of the phase velocity. The force due to gravity has been neglected, apart from its effect on stratification according to density.

We will be interested in the reflection of sound at a stratified layer between two uniform media (1 and 2), the properties of the interfacial layer being characterized by a density $\rho(z)$ and a velocity $c(z)$, where z is the depth (planar stratification is assumed). For a plane monochromatic wave propagating in the zx plane, solutions of (1) have the form

$$p_a(z,x,t) = e^{i(Kx-\omega t)}P(z), \tag{2}$$

where ω is the angular frequency of the wave, and K is the x component of the wave vector, which is a constant of the motion:

$$K = (\omega/c_1)\sin\theta_1 = (\omega/c_2)\sin\theta_2. \tag{3}$$

Here c_1 and c_2 are the phase velocities in the two bounding media, and θ_1 and θ_2 are the angles of incidence and refraction [equivalently, (3) can be read as the invariance of c^{-1} times the cosine of the grazing angle]. The differential equation for $P(z)$ can be put in the form

$$\rho\frac{d}{dz}\left(\frac{1}{\rho}\frac{dP}{dz}\right) + q^2 P = 0, \tag{4}$$

where $q(z)$ is defined by

$$q^2(z) = \omega^2/c^2(z) - K^2. \tag{5}$$

From (3) and (5), the limiting values of q, the normal component of the wave vector, are $q_1 = (\omega/c_1)\cos\theta_1$, $q_2 = (\omega/c_2)\cos\theta_2$. As noted in Ref. 1 (Sec. 1-4), Eq. (4) has the same form as that for the electromagnetic p wave, which may be reduced to simpler form by transformation of the independent variable.[6] The analogous reduction for acoustic waves is obtained by introducing a new depth variable dilated in proportion to the local value of the density, $dZ = \rho\, dz$:

$$\frac{d^2P}{dZ^2} + Q^2 P = 0, \quad Q = \frac{q}{\rho}. \tag{6}$$

The same approach is taken by Godin.[7] Our main use of (6) here is to motivate the introduction of Q as an effective normal component of the wave vector.

The reflection and transmission amplitudes r and t are defined in terms of the limiting forms of the acoustic pressure in media 1 and 2:

$$e^{iq_1z} + r\, e^{-iq_1z} \leftarrow P(z) \rightarrow t\, e^{iq_2z}. \quad (7)$$

In the case of a sharp transition (an interface of zero thickness) at $z = 0$, the continuity of P and dP/dZ, implied by (4) or (6), gives

$$r_0 = (Q_1 - Q_2)/(Q_1 + Q_2), \quad t_0 = 2Q_1/(Q_1 + Q_2). \quad (8)$$

From Snell's law (3) and the definitions of q and Q, $Q = K/\rho \tan\theta$, so the reflection and transmission amplitudes may be written as

$$r_0 = \frac{\rho_2 \tan\theta_2 - \rho_1 \tan\theta_1}{\rho_2 \tan\theta_2 + \rho_1 \tan\theta_1}, \quad t_0 = \frac{2\rho_2 \tan\theta_2}{\rho_2 \tan\theta_2 + \rho_1 \tan\theta_1}. \quad (9)$$

These familiar results go back to Green[8]; they are rederived here to introduce the notation, and because we will show that, under certain conditions on the ρ and c profiles, $|r| < |r_0|$.

II. AN IDENTITY FOR THE REFLECTION AMPLITUDE

The method by which the linear second-order differential equation like (4) can be transformed into a nonlinear first-order equation for a quantity proportional to the reflection amplitude is well known: See, for example, Ref. 3, Sec. 17 or Ref. 1, Chap. 5. We give only an outline here and then derive the identity relevant to the problem at hand.

Equation (4) can be written as a pair of coupled linear equations in P and $D = dP/\rho\, dz = dP/dZ$. We set

$$P = F + G, \quad D = iQ(F - G), \quad (10)$$

eliminate P and D, and obtain a pair of coupled equations for F and G:

$$F' = iqF - (Q'/2Q)(F - G), \quad G' = -iqG + (Q'/2Q)(F - G). \quad (11)$$

Here and in what follows, the prime denotes differentiation with respect to z. Note that, in uniform media, $F \sim e^{iqz}$ and $G \sim e^{-iqz}$. In medium 1, therefore, the ratio $\tilde{r} = G/F$ is equal to $\exp(-2iq_1z)r$. For the reflection problem, in which there is no wave traveling toward the interface from medium 2, $G = 0$ in medium 2 and

$$e^{-2iq_1z} r \leftarrow \tilde{r} \rightarrow 0. \quad (12)$$

An equation for $\tilde{r}$ is obtained from (11) by multiplying the first equation by G/F^2, the second by $1/F$, and subtracting one from the other. The result is the Riccati-type equation

$$\tilde{r}' + 2iq\tilde{r} - (Q'/2Q)(1 - \tilde{r}^2) = 0. \quad (13)$$

This equation remains valid in the presence of attenuation (when q and Q become complex), but the following equations require modification.

Our interest is mainly in the absolute magnitude of the reflection amplitude, $|r|$, and in the reflectivity $R = |r|^2$. We therefore set $r = |r|e^{i\psi}$. The real and imaginary parts of (13) are

$$|\tilde{r}|' = (Q'/2Q)(1 - |\tilde{r}|^2)\cos\psi, \quad (14)$$

$$\psi' + 2q = -(Q'/2Q)(|\tilde{r}| + |\tilde{r}|^{-1})\sin\psi. \quad (15)$$

Equation (14) may be written as

$$|\tilde{r}|'\left(\frac{1}{1 - |\tilde{r}|} + \frac{1}{1 + |\tilde{r}|}\right) = \frac{Q'}{Q}\cos\psi, \quad (16)$$

integration of which from $z = -\infty$ to $+\infty$, with the use of (12), gives the identity

$$\log\frac{1 + |r|}{1 - |r|} = \int_{-\infty}^{\infty} dz \frac{Q'}{Q}\cos\psi. \quad (17)$$

III. AN UPPER BOUND FOR THE REFLECTIVITY

Let α be the dimensionless quantity on the right side for (17). Then $|r| = \tanh \alpha/2$ and

$$R = \tanh^2 \alpha/2. \quad (18)$$

Clearly $R \leqslant 1$, a result true for reflection from any passive medium (see, for example, Sec. 2-2 of Ref. 1). This physically necessary upper bound can be much improved in certain cases.

Suppose that Q'/Q has one sign throughout the stratification, which implies that Q increases or decreases monotonically. [In the absence of absorption, $Q = q/\rho$ is either real or imaginary: When $c_2 > c_1$, it is imaginary for $\theta_1 > \theta_c = \arcsin(c_1/c_2)$ and there is total internal reflection.] Since the cosine of the (unknown) phase ψ is bounded above by $+1$ and below by -1, the right side of (17) is bounded above by $\log(Q_{max}/Q_{min})$, where Q_{max} and Q_{min} are the greater and lesser of Q_1 and Q_2. Thus

$$(1 + |r|)/(1 - |r|) \leqslant Q_{max}/Q_{min} \quad (19)$$

and

$$R = |r|^2 \leqslant [(Q_1 - Q_2)/(Q_1 + Q_2)]^2 = R_0 \quad (20)$$

if Q is monotonic. Thus a profile for which Q is monotonic will not reflect more than an abrupt transition between the same two media, at the same angle of incidence, and at any frequency. (The reflectance from a step profile is independent of frequency.)

Under what circumstances is Q monotonic? From (5) and $Q = q/\rho$,

$$Q^2(z) = [\omega^2/c^2(z) - K^2]/\rho^2(z). \quad (21)$$

From (21) we find

$$\frac{dQ^2}{dz} = -\frac{2}{\rho^2}\left(q^2 \frac{d\log(\rho c)}{dz} + K^2 \frac{d\log c}{dz}\right). \quad (22)$$

Normal incidence ($K=0$): dQ^2/dz has the sign of $-d\log(\rho c)/dz$, so if ρc increases or decreases monotonically, the reflectivity at normal incidence is never greater than that for a sharp interface.

General incidence: If ρc and c both increase or both decrease monotonically, the reflectivity at *any angle* will be smaller than the reflectivity (at the same angle) at an abrupt change between the same bounding media. [In this case, there is no Green's angle, at which $R_0 = 0$; see Ref. 1, Eq. (1.61).] If, on the other hand, ρc increases monotonically and c decreases monotonically (or vice versa), the expression within large parentheses in (22) may change sign, in which case there is the possibility of greater reflection than from a sharp interface.

IV. THE RAYLEIGH APPROXIMATION

The nonlinear first-order equation (13) gives $\tilde{r}(z)$, where the limit as $z \to -\infty$ of $\exp(2iq_1 z)\tilde{r}$ is the reflection amplitude. It is possible to obtain a Riccati-type equation for $r(z)$, with $r(-\infty)$ being the reflection amplitude itself. Instead of the substitutions (10), we write

$$P = f e^{i\phi} + g e^{-i\phi}, \tag{23}$$

$$D = iQ(f e^{i\phi} - g e^{-i\phi}), \tag{24}$$

where ϕ (known as the phase integral) is defined by

$$\phi(z) = \int^{z} d\xi\, q(\xi). \tag{25}$$

Since the original equation (4) can be written as two simultaneous equations in P and D, namely,

$$\rho D' + q^2 P = 0, \quad D = P'/\rho, \tag{26}$$

we have two equations for the two unknown functions f and g. Solving these gives

$$f' + (Q'/2Q)(f - g e^{-2i\phi}) = 0, \tag{27}$$

$$g' + (Q'/2Q)(g - f e^{2i\phi}) = 0. \tag{28}$$

We see that f and g are constant where Q is constant, in particular at $z = -\infty$ (in the uniform medium 1). We choose the normally unspecified lower limit in (25) to be such that

$$\phi(z) \to q_1 z \quad \text{as} \quad z \to -\infty. \tag{29}$$

In the same limit, this makes P tend to

$$f(-\infty) e^{iq_1 z} + g(-\infty) e^{-iq_1 z}. \tag{30}$$

Thus, if we define a reflection function $r(z) \equiv g(z)/f(z)$, $r(-\infty)$ is the reflection amplitude, since this is, by definition, the ratio of the coefficient of $e^{-iq_1 z}$ to that of $e^{iq_1 z}$. To obtain an equation for $r(z)$, we take g times (27) minus f times (28) and divide the result by f^2. The result is the nonlinear first-order equation

$$r'(z) = (Q'/2Q)[e^{2i\phi} - r^2(z) e^{-2i\phi}]. \tag{31}$$

In the reflection problem, $g(\infty) = 0$ and thus $r(\infty) = 0$. Integration of (31) from $-\infty$ to $+\infty$ thus gives

$$r \equiv r(-\infty) = -\int_{-\infty}^{\infty} dz \frac{Q'}{2Q} [e^{2i\phi} - r^2(z) e^{-2i\phi}]. \tag{32}$$

This is an exact result for the reflection amplitude, which remains valid in the presence of attenuation in the stratification. The Rayleigh (or weak-reflection) approximation is obtained by setting $r(z) = 0$ on the right of (32):

$$r_R = -\int_{-\infty}^{\infty} dz \frac{Q'}{2Q} e^{2i\phi}. \tag{33}$$

An explicit expression for the Rayleigh reflectivity can be obtained in the long-wave limit, where ϕ is nearly constant over the stratification and

$$R_R = |r_R|^2 \to [\tfrac{1}{2}\log(Q_1/Q_2)]^2. \tag{34}$$

Since $(\frac{1}{2}\log x)^2 \geqslant (x-1)^2/(x+1)^2$ for positive x, the Rayleigh reflectivity is not less than the Green (or sharp transition) value:

$$R_0 = [(Q_1 - Q_2)/(Q_1 + Q_2)]^2 \tag{35}$$

in the long-wave limit.

The correction to the Rayleigh approximation reflection amplitude is the second term in (32); namely,

$$\Delta r = \int_{-\infty}^{\infty} dz \frac{Q'}{2Q} r^2(z) e^{2i\phi}. \tag{36}$$

The physical meaning of $r(z)$ is that of the reflection amplitude of a stratification "truncated" at z [see Sec. 5-1 of Ref. 1 for an interpretation of the related $r(z)$]. Thus the Rayleigh approximation can be expected to be accurate if the reflection from any truncation of the stratification is weak. This is so particularly in the case of smoothly varying properties in the short-wave (high-frequency) limit.

V. A SIMPLE EXAMPLE: REFLECTION BY A UNIFORM LAYER

We will compare the results of the previous sections with the exact reflectivity of a uniform layer of thickness Δz, density ρ, and speed of sound c, between uniform media 1 and 2. The reflection amplitude may be obtained by matching solutions made up of $\exp(\pm iqz)$ to $\exp(iq_1 z) + r\exp(-iq_1 z)$ in medium 1, and $t\exp(iq_2 z)$ in medium 2, using the continuity of P and P'/ρ. The result is (cf. Ref. 3, Eq. 5.10)

$$r = e^{2iq_1 z_1} \frac{r_1 + r_2 e^{2iq\Delta z}}{1 + r_1 r_2 e^{2iq\Delta z}}, \tag{37}$$

where z_1 and z_2 are the boundaries of the uniform layer, $\Delta z = z_2 - z_1$ is its thickness, and r_1, r_2 are the Green reflection coefficients at the two boundaries:

$$r_1 = (Q_1 - Q)/(Q_1 + Q), \qquad r_2 = (Q - Q_2)/(Q + Q_2). \tag{38}$$

The reflectivity is the absolute square of (37); for real q_1, q, and q_2, this is

$$R = |r|^2 = \frac{r_1^2 + 2r_1 r_2 \cos 2q\Delta z + r_2^2}{1 + 2r_1 r_2 \cos 2q\Delta z + r_1^2 r_2^2}. \tag{39}$$

For fixed frequency and angle of incidence (fixed q and Q's), this is a periodic function of the layer thickness Δz, with period π/q (equal to $\pi c/\omega = \lambda/2$ at normal incidence), provided q is real. The extrema of (39) occur when $\cos 2q\Delta z = \pm 1$; these values are

$$R_+ = \left(\frac{Q_1 - Q_2}{Q_1 + Q_2}\right)^2 = R_0, \quad R_- = \left(\frac{Q_1 Q_2 - Q^2}{Q_1 Q_2 + Q^2}\right). \tag{40}$$

The theorem of Sec. III states that if $Q(z)$ is monotonic, $R \leqslant R_0$. Applied to the problem at hand, this reads that if Q lies between Q_1 and Q_2, the reflectance must be no greater than R_0. The implication is that R_+ is greater than R_- when the value of Q is between Q_1 and Q_2. From (40), we find that R_+ is greater than R_- when

$$(Q^2 - Q_1^2)(Q^2 - Q_2^2) < 0, \tag{41}$$

which is true when Q lies between Q_1 and Q_2, in agreement

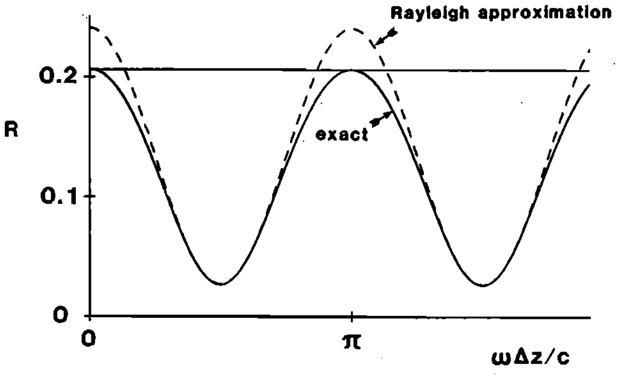

FIG. 1. Normal incidence reflectivity from a uniform layer. The solid curve is the exact reflectivity [Eq. (39)]; the dashed curve is the Rayleigh approximation [Eq. (44)]. The density values used (in g/cm^3) are $\rho_1 = 1$, $\rho = 1.7$, $\rho_2 = 2$; the corresponding sound speeds (in km/s) are $c_1 = 1.5$, $c = 1.7$, $c_2 = 2$. The horizontal line shows the upper bound derived in Sec. III.

with the theorem.

Finally, we will compare the exact reflectivity with that obtained from the Rayleigh approximation of Sec. IV. We rewrite (33) as

$$r_R = -\frac{1}{2}\int_{-\infty}^{\infty} dz\, e^{2i\phi} \frac{d}{dz}(\log Q). \qquad (42)$$

At z_1 and z_2, $\log Q$ is discontinuous, by the amounts $\log(Q/Q_1)$ and $\log(Q_2/Q)$. The derivative of a step function is a delta function of strength equal to the magnitude of the step, so, for the uniform layer under consideration,

$$r_R = -\frac{1}{2}\left(e^{2i\phi_1}\log\frac{Q}{Q_1} + e^{2i\phi_2}\log\frac{Q_2}{Q}\right), \qquad (43)$$

and the Rayleigh approximation reflectance is

$$R_R = \frac{1}{4}\left[\left(\log\frac{Q}{Q_1}\right)^2 + \left(\log\frac{Q_2}{Q}\right)^2 + 2\log\frac{Q}{Q_1} \times \log\frac{Q_2}{Q}\cos 2q\,\Delta z\right]. \qquad (44)$$

(The argument of the cosine follows from $\phi_2 - \phi_1 = q\Delta z$.) The agreement with the exact result (39) is good when $|r_1|$ and $|r_2|$ are small, since then $\frac{1}{2}\log(Q_1/Q) \approx (Q_1 - Q)/(Q_1 + Q)$ and $\frac{1}{2}\log(Q/Q_2) \approx (Q - Q_2)/(Q + Q_2)$. To estimate the error in the first of these relations, set $Q_1 = Q + \delta Q$. We find that the two sides agree to order $(\delta Q)^2$, the difference $\frac{1}{2}\log(Q_1/Q) - (Q_1 - Q)/(Q_1 + Q)$ having the leading term $(\delta Q/Q)^3/24$.

Figure 1 compares the normal incidence exact and approximate reflectivities, as a function of the thickness of the layer. The parameters are chosen to approximate a layer of sediment on a seafloor or lake bottom.[9] We note that the Rayleigh reflectivity is most accurate where the reflection is smallest.

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