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Angular momentum of sound pulses

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Abstract

Three-dimensionally localized acoustic pulses in an isotropic fluid medium necessarily have transverse components of momentum density. Those with an azimuthal component of momentum density can carry angular momentum. The component of total pulse angular momentum along the direction of the total momentum is an invariant (constant in time and independent of choice of origin). The pulse energy, momentum and angular momentum are evaluated analytically for a family of localized solutions of the wave equation. In the limit where the pulses have many oscillations within their spatial extent ($ka \gg 1$, where k is the wavenumber and a determines the size of a pulse), the energy, momentum and angular momentum are consistent with a multiphonon representation of the pulse, each phonon having energy $\hbar ck$, momentum $\hbar k$ and angular momentum $\hbar m$ (with integer m).

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The idea of light carrying angular momentum goes back to Poynting in 1909 [1]. The first measurement of the torque exerted by a beam of polarized light was made by Beth [2]. The field of optical angular momentum is still very active: for a selection of reprints, see [3]. In contrast, acoustic angular momentum is not even mentioned in standard texts [4–10]. This is surprising, given that a common source of sound, in air and in water, is the propeller. The rotation of the propeller imparts a torque to the fluid medium, and we might well expect the radiated sound to have angular momentum. Although acoustical torques on objects have been studied and used since Rayleigh's proposal in 1882 [11–14], the sound beams causing the torques did not carry intrinsic angular momentum. The only existing discussion of intrinsic acoustic angular momentum appears to be for 'helicoidal' acoustical beams, studied and produced by Hefner and Marston [15]. The same types of transducer used to produce helicoidal beams could also produce the angular-momentum-carrying pulses studied here, and such pulses could be used to give impulsive torques, for example to rotating free drops in the microgravity environment of low earth orbit [14].

The purpose of this note is to define the angular momentum content of a sound pulse, show that it is conserved when viscous damping and scattering are neglected, and present some simple solutions of the wave equation, the sound associated with which carries angular momentum. The results are compared with that peculiar quantum ghost, the textbook phonon, which is infinite in extent in space and time, yet carries finite energy $\hbar ck$ and finite momentum $\hbar k$ (where c is the speed of sound). We shall show by explicit calculation that, for a class of three-dimensionally localized pulses, the energy, momentum and angular momentum can have quite complex dependences on the wavenumber k. However, for $ka \gg 1$ (where a determines the size of the pulse) the expressions simplify, and in this limit there is agreement with the phonon picture. Moreover, in this limit one can associate angular momentum $m\hbar$ with each phonon, m being an integer.

2. Angular momentum of an acoustic pulse

We have shown previously that the Landau–Lifshitz [5] energy and momentum densities $e(\mathbf{r}, t)$ and $\mathbf{p}(\mathbf{r}, t)$ give total pulse energy and momentum

$$E = \int \mathrm{d}^3 r\, e(\mathbf{r}, t), \qquad \mathbf{P} = \int \mathrm{d}^3 r\, \mathbf{p}(\mathbf{r}, t) \tag{1}$$

which are independent of time [16]. (That pulse energy and momentum are conserved just verifies the consistency of the theory, in which viscosity and scattering are assumed to be negligible.) The energy and momentum densities are expanded in powers of the velocity potential $V(\mathbf{r}, t)$ which determines the curl-free first-order fluid velocity

$$\mathbf{v}_1(\mathbf{r}, t) = \nabla V(\mathbf{r}, t). \tag{2}$$

The velocity potential satisfies the wave equation

$$\nabla^2 V = \partial_{ct}^2 V, \tag{3}$$

as do the first-order fluid density

$$\rho_1 = -(\rho_0/c)\partial_{ct} V \tag{4}$$

and the first-order pressure $p_1 = c^2 \rho_1$.

The first-order energy and momentum densities are

$$e_1 = (e_0 + p_0)\rho_0^{-1}\rho_1 = -(e_0 + p_0)c^{-1}\partial_{ct} V, \qquad \mathbf{p}_1 = \rho_0 \mathbf{v}_1 = \rho_0 \nabla V. \tag{5}$$

If the conservation of matter is assumed to hold at each order, then $\int \mathrm{d}^3 r\, \rho_1 = 0$. Also, $\int \mathrm{d}^3 r\, \nabla V = 0$ for a pulse of finite extent in all three dimensions. Thus, when

$$\int \mathrm{d}^3 r\, \partial_{ct} V = 0, \tag{6}$$

the first-order energy and momentum densities give zero contribution to the pulse total energy and total momentum, which become [5, 16]

$$\begin{aligned} E &= \tfrac{1}{2}\rho_0 \int \mathrm{d}^3 r \left[(\partial_{ct} V)^2 + (\nabla V)^2\right] \\ c\mathbf{P} &= -\rho_0 \int \mathrm{d}^3 r\, (\partial_{ct} V)\nabla V. \end{aligned} \tag{7}$$

(Landau and Lifshitz omit the second-order momentum density term $\rho_0 \mathbf{v}_2$, but it is shown in [16] that $\mathbf{v}_2$, like $\mathbf{v}_1$, is irrotational and thus expressible as the gradient of a potential, so this term also contributes zero to the total pulse momentum.)

We now consider the angular momentum of the pulse. The angular momentum density is

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_1 + \rho_0 \mathbf{v}_2 + \cdots), \tag{8}$$

and the total angular momentum is

$$\mathbf{J} = \int \mathrm{d}^3 r \, \mathbf{j} = \int \mathrm{d}^3 r \, \mathbf{r} \times \mathbf{p}. \tag{9}$$

In a translation of the coordinate system, $\mathbf{r} \to \mathbf{r} - \mathbf{a}$, the total angular momentum transforms to $\mathbf{J} \to \mathbf{J} - \mathbf{a} \times \mathbf{P}$. Thus the component of $\mathbf{J}$ parallel to $\mathbf{P}$ is invariant to the choice of origin. We shall take the direction of $\mathbf{P}$ to define the z-axis; then J_z is the invariant component of interest. The first-order contribution to J_z is proportional to the integral over all space of

$$(\mathbf{r} \times \nabla V)_z = (x\partial_y - y\partial_x)V = \partial_\phi V \tag{10}$$

where ϕ is the azimuthal angle. Thus the first-order contribution is zero. Likewise, the $\rho_0 \mathbf{v}_2$ term of the momentum density contributes zero, since $\mathbf{v}_2$ is also the gradient of a potential. We are left with

$$J_z = \int \mathrm{d}^3 r \, \rho_1 \partial_\phi V = -\frac{\rho_0}{c} \int \mathrm{d}^3 r \, (\partial_{\mathrm{ct}} V)(\partial_\phi V). \tag{11}$$

We see immediately that the velocity potential must depend on the azimuthal angle for non-zero j_z and J_z.

Let us suppose that V does depend on ϕ, and look at the time-dependence of the resulting angular momentum. We have

$$-\frac{c}{\rho_0} \partial_{\mathrm{ct}} J_z = \int \mathrm{d}^3 r \, \left[(\partial_{\mathrm{ct}}^2 V)(\partial_\phi V) + (\partial_{\mathrm{ct}} V)(\partial_{\mathrm{ct}} \partial_\phi V) \right]. \tag{12}$$

Integration over ϕ of the last term, which equals $\frac{1}{2}\partial_\phi(\partial_{\mathrm{ct}} V)^2$, gives zero. In the first term, we can replace $\partial_{\mathrm{ct}}^2 V$ by $\nabla^2 V$, since the velocity potential satisfies the wave equation. In cylindrical polar coordinates, the right-hand side of (12) then becomes

$$\int_{-\infty}^{\infty} \mathrm{d}z \int_0^{\infty} \mathrm{d}\rho \, \rho \int_0^{2\pi} \mathrm{d}\phi (\partial_\phi V) \left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2 \right) V. \tag{13}$$

The derivatives in ρ contribute $(\rho\partial_\rho^2 + \partial_\rho)V = \partial_\rho(\rho\partial_\rho V)$ to the integrand, and integration by parts with respect to ρ gives us

$$\int_0^{\infty} \mathrm{d}\rho (\partial_\phi V)\partial_\rho(\rho\partial_\rho V) = -\int_0^{\infty} \mathrm{d}\rho (\partial_\rho \partial_\phi V)\rho\partial_\rho V = -\tfrac{1}{2}\int_0^{\infty} \mathrm{d}\rho \, \rho \partial_\phi (\partial_\rho V)^2 \tag{14}$$

which gives zero on integration over ϕ. Likewise, we have

$$\int_{-\infty}^{\infty} \mathrm{d}z (\partial_\phi V)(\partial_z^2 V) = -\int_{-\infty}^{\infty} \mathrm{d}z (\partial_z \partial_\phi V)(\partial_z V) = -\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}z \, \partial_\phi (\partial_z V)^2 \tag{15}$$

which again gives zero on integration over ϕ. Finally, $(\partial_\phi V)(\partial_\phi^2 V) = \frac{1}{2}\partial_\phi(\partial_\phi V)^2$, and this also integrates to zero over ϕ.

Thus $\partial_{\mathrm{ct}} J_z = 0$, and the total angular momentum of the pulse is conserved. This must hold for self-consistency of the formalism, since dissipative effects have been neglected. We shall see, however, that, as the pulse propagates, the spatial distribution of angular momentum changes.

3. Localized solutions of the wave equation with azimuthal dependence

Hillion [17] found a family of localized solutions of the wave equation. Let $f(s)$ be any twice-differentiable function of the variable

$$s = \frac{x^2 + y^2}{b + \mathrm{i}(z + ct)} - \mathrm{i}(z - ct). \tag{16}$$

Then $f(s)/[b + \mathrm{i}(z + ct)]$ satisfies the wave equation (3). An alternative route to this result was given in [18]. Here we need extensions of this family of solutions, incorporating azimuthal dependence, since, for acoustic pulses, the angular momentum density is proportional to $\partial_\phi V$. (The situation is different for electromagnetic pulses, where a solution of the wave equation which is independent of ϕ can have non-zero j_z and J_z [18].) One such extension was given in [19], namely that for positive integer m

$$\psi_m^\pm = \left[\frac{x \pm \mathrm{i}y}{b + \mathrm{i}(z + ct)}\right]^m \frac{f(s)}{b + \mathrm{i}(z + ct)} = \left[\frac{\rho}{b + \mathrm{i}(z + ct)}\right]^m \mathrm{e}^{\pm \mathrm{i}m\phi} \frac{f(s)}{b + \mathrm{i}(z + ct)} \tag{17}$$

is a solution of the wave equation, where f is again any twice-differentiable function, and $\rho = (x^2 + y^2)^{\frac{1}{2}}$ is the distance from the z-axis.

Consider the choice $f(s) = \mathrm{e}^{-ks}/(s + a)$. Then the set $\psi_m^\pm$ is characterized by an integer m, a wavenumber k, and lengths a and b. (We shall assume k, a and b are all real and positive.) With this choice of f, the pulse wavefunctions become

$$\psi_m^\pm = \left[\frac{\rho}{b + \mathrm{i}(z + ct)}\right]^m \mathrm{e}^{\pm \mathrm{i}m\phi} \frac{\mathrm{e}^{-ks}}{\rho^2 + [a - \mathrm{i}(z - ct)]\,[b + \mathrm{i}(z + ct)]}. \tag{18}$$

For $m = 0$, we have a solution of the wave equation which tends to the plane wave $\mathrm{e}^{\mathrm{i}k(z-ct)}$ in a region near the z-axis where ρ^2 is small compared to ab, $k\rho^2$ is small compared to b, and also a and b are large compared to $|z - ct|$ and $|z + ct|$, respectively. For non-zero m, the wavefunction is zero on the z-axis, which leads to energy and momentum densities peaking off-axis, as we shall see.

4. Energy, momentum and angular momentum of a family of pulses

We shall find the total energy, total momentum, and total angular momentum of acoustic pulses derived from two subfamilies of (18), namely those with $a = b$ and $m = 1$ and 2. Those with higher m have similar but more complicated structures, and we shall give only the $ka \gg 1$ limit. The total energy, momentum and angular momentum are given by (7) and (11), with the velocity potential given by

$$V_m^\pm(\rho, z, \phi, t) = \left[\frac{\rho}{a + \mathrm{i}(z + ct)}\right]^m \frac{\mathrm{e}^{\pm \mathrm{i}m\phi}(ka)^{1+m/2} a^2 V_0 \exp\left[-\frac{k\rho^2}{a+\mathrm{i}(z+ct)} + \mathrm{i}k(z - ct)\right]}{\rho^2 + [a - \mathrm{i}(z - ct)]\,[a + \mathrm{i}(z + ct)]}. \tag{19}$$

We have inserted the factor $a^2 V_0$ to make V_0 carry the magnitude and dimension (speed $\times$ distance) of the velocity potential, and the factor $(ka)^{1+m/2}$ to connect with the phonon picture (see below). Apart from V_0, a and m, the solutions are characterized by the dimensionless parameter ka. Integration over ϕ of $\partial_{ct} V$ gives zero, so condition (6) is satisfied.

In carrying out the integrations in (7) and (11) we first note that we can set $t = 0$, since E, $\mathbf{P}$ and $\mathbf{J}$ are independent of time. Second, in taking either the real or imaginary parts of (19) as the velocity potential, the azimuthal dependence is bilinear in $\cos(m\phi)$ and $\sin(m\phi)$, since the velocity potential and its derivatives are linear in $\cos(m\phi)$ and $\sin(m\phi)$. The average over ϕ is thus given by replacing $\cos^2(m\phi)$ and $\sin^2(m\phi)$ by $\frac{1}{2}$, and $\cos(m\phi)\sin(m\phi)$ by zero.

Considering the energy, for example, with $\bar{e}$ representing the above substitutions in the energy density $e(\rho, z, \phi, t)$, we have

$$E = 2\pi \int_{-\infty}^{\infty} \mathrm{d}z \int_{0}^{\infty} \mathrm{d}\rho\, \rho\, \bar{e}(\rho, z). \tag{20}$$

We find it advantageous to work in terms of the dimensionless quantities

$$\alpha = ka, \qquad \eta = \frac{\rho^2}{a^2 + z^2} + 1, \qquad \zeta = kz. \tag{21}$$

The integration in (20) then becomes

$$E = \pi \int_{-\infty}^{\infty} \mathrm{d}z\,(a^2 + z^2) \int_{1}^{\infty} \mathrm{d}\eta\, \bar{e} = \pi k^{-3} \int_{-\infty}^{\infty} \mathrm{d}\zeta\,(\alpha^2 + \zeta^2) \int_{1}^{\infty} \mathrm{d}\eta\, \bar{e} \tag{22}$$

with $\bar{e} = \rho_0 a^4 V_0^2 k^6 (ka)^{|m|} \varepsilon(\alpha, \eta, \zeta)$. For example, when $m = \pm 1$ and either the real or the imaginary parts of (19) are used as the velocity potential, the dimensionless function ε is given by

$$\varepsilon = \frac{\alpha^2(\eta - 1)\eta^3 - 2\alpha\eta(\eta - 1)^2 + \eta^2 + 2\eta - 2 + \eta^3(\eta - 1)\zeta^2}{2\eta^3(\alpha^2 + \zeta^2)\mathrm{e}^{2\alpha(\eta-1)}}. \tag{23}$$

The exponential factor in (23) assures the convergence of the η integral, which can be evaluated in terms of exponential integrals Ei. The remaining integration over ζ is elementary. We find

$$E_1^{\pm} = \frac{\pi^2}{8} \rho_0 a V_0^2 \alpha\, [1 + 2F(2\alpha)] \tag{24}$$

where the function $F(x)$ is defined for positive x by

$$F(x) = \mathrm{e}^x \mathrm{Ei}(1, x) = -\mathrm{e}^x \mathrm{Ei}(-x) = -\mathrm{e}^x \int_{-\infty}^{-x} \mathrm{d}y\, y^{-1} \mathrm{e}^y = \mathrm{e}^x \int_{x}^{\infty} \mathrm{d}y\, y^{-1} \mathrm{e}^{-y} \tag{25}$$

$F(x)$ is logarithmic in x for small positive x,

$$F(x) = -\ln x - \gamma + (-\ln x - \gamma + 1)x + \mathrm{O}(x^2). \tag{26}$$

For large x, the asymptotic expansion is

$$F(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \mathrm{O}(x^{-5}). \tag{27}$$

Results for the total momentum and angular momentum are found similarly. They are

$$\begin{aligned} cP_{1z}^{\pm} &= \frac{\pi^2}{8} \rho_0 a V_0^2 \alpha\, [1 - 2F(2\alpha)] \\ ckJ_{1z}^{\pm} &= \pm \frac{\pi^2}{8} \rho_0 a V_0^2 2\alpha^2 F(2\alpha). \end{aligned} \tag{28}$$

Note that P_{1z} is negative for small $\alpha = ka$: the pulse is predominantly forward-propagating only for $\alpha \gtrsim 0.644\,54$. For large α, we see from (27) that $E_1^{\pm}$ and $cP_{1z}^{\pm}$ tend to the common value $\frac{\pi^2}{8}\rho_0 ka^2 V_0^2$, while $ckJ_{1z}^{\pm}$ tends to $\pm$ this value. This result is in accord with the pulse being composed of N phonons, each of energy $\hbar ck$, momentum $\hbar k$, and angular momentum $\pm\hbar$, where

$$N \approx \frac{E_1^{\pm}}{\hbar ck} \approx \frac{P_{1z}^{\pm}}{\hbar k} \approx \frac{J_{1z}^{\pm}}{\pm\hbar} \approx \frac{\pi^2}{8} \frac{\rho_0 a^2 V_0^2}{\hbar c}. \tag{29}$$

A similar interpretation is possible in the $m = 0$ case, which is examined in [20].

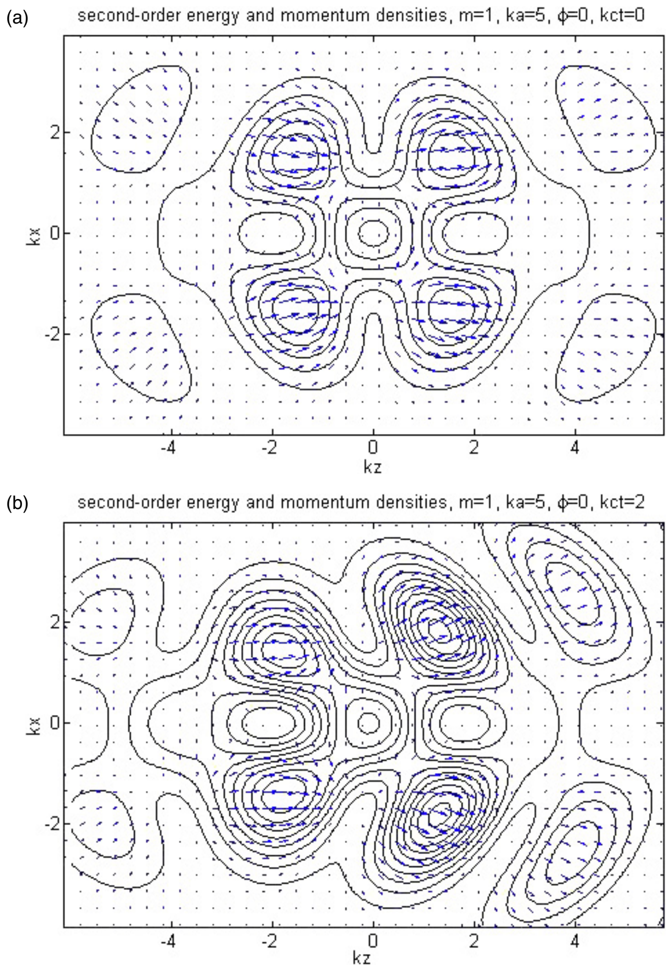

Figure 1. Second-order energy density (contours) and momentum density (arrows) in the *zx* plane. Part (a) is at zero time, part (b) at $kct = 2$. Note that the energy maxima along the propagation axis are not associated with momentum maxima, which are off-axis.

For $m = \pm 2$, the total energy, momentum and angular momentum can be obtained in the same way. We find

$$
\begin{aligned}
E_2^{\pm} &= \frac{\pi^2}{8}\rho_0 a V_0^2 \alpha\,[3 - 4\alpha(2\alpha)] \\
cP_{2z}^{\pm} &= \frac{\pi^2}{8}\rho_0 a V_0^2 \alpha\,[8\alpha F(2\alpha) - 3] \\
ckJ_{2z}^{\pm} &= \pm\frac{\pi^2}{2}\rho_0 a V_0^2 \alpha^2\,[1 - 2\alpha F(2\alpha)]\,.
\end{aligned}
\tag{30}
$$

Again the total momentum is negative for small α, this time up to $\alpha \approx 1.181\,94$. At large α, the energy and c times the momentum tend to the common value $\frac{\pi^2}{8}\rho_0 k a^2 V_0^2$ as before, but now $ckJ_{2z}^{\pm}$ tends to $\pm$ twice this value. We can therefore regard the $ka \gg 1$ pulse as being made up of N phonons, each of energy $\hbar ck$, momentum $\hbar k$, and angular momentum $\pm 2\hbar$, where N

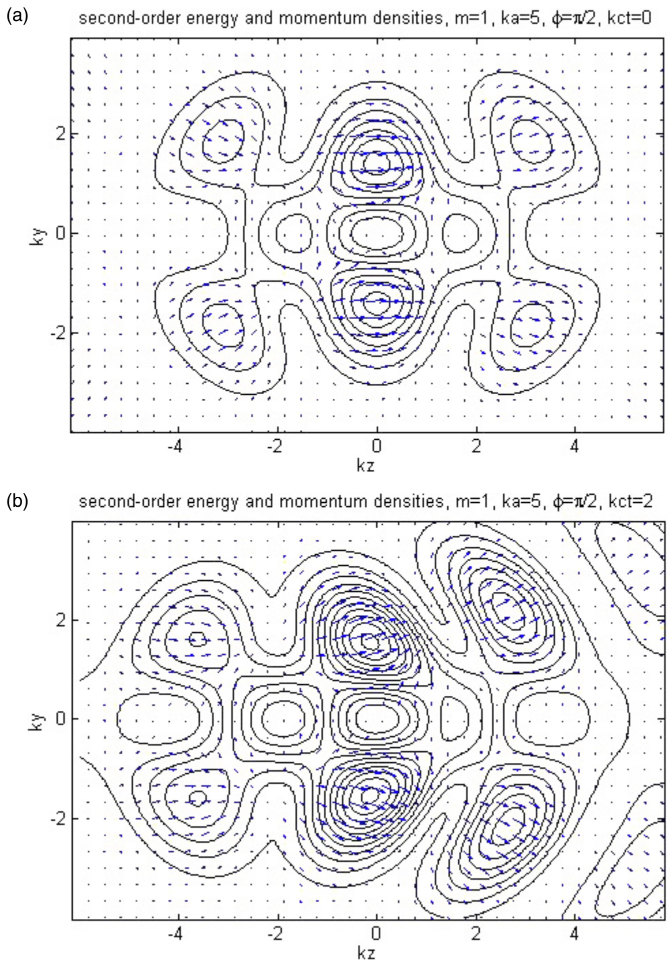

Figure 2. As for figure 1, but now viewed in the zy plane, i.e. at $\phi = \pi/2$. Note that the main momentum maxima are at $z = 0$, in contrast to figure 1, where $\phi = 0$.

takes the same value as in (29):

$$N \approx \frac{E_2^{\pm}}{\hbar ck} \approx \frac{P_{2z}^{\pm}}{\hbar k} \approx \frac{J_{2z}^{\pm}}{\pm 2\hbar} \approx \frac{\pi^2}{8}\frac{\rho_0 a^2 V_0^2}{\hbar c}. \tag{31}$$

For $ka \gg 1$, we find, for general m,

$$E^{\pm}, cP_z^{\pm}, ckJ_z^{\pm}/m \to \pi^2 \frac{m!}{2^{m+2}} \rho_0 k a^2 V_0^2. \tag{32}$$

Note that the multiphonon picture of an acoustic pulse works only in the limit of large ka. It is interesting (and apparently a new result) that in this limit one can associate an angular momentum $m\hbar$ with each phonon.

5. The helical nature of the pulses

We shall give plots of the second-order energy, momentum and angular momentum densities to illustrate the $m = +1$ pulse. All the plots are for the moderately large value $ka = 5$ (for

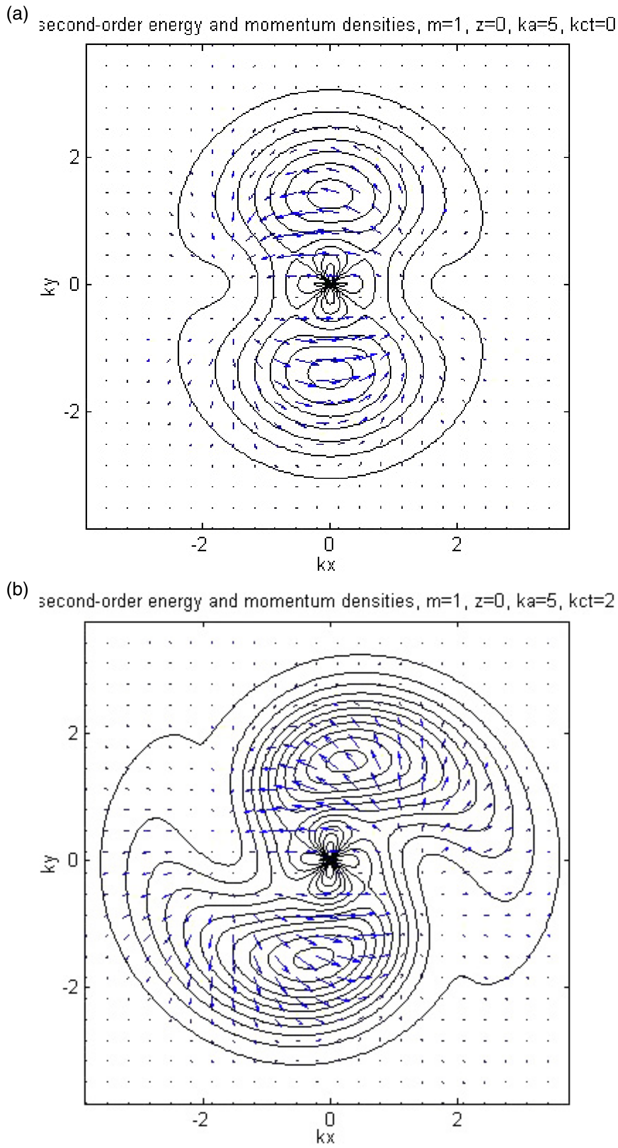

Figure 3. Second-order energy and momentum densities in a transverse section of the pulse (the plane $z = 0$). The angular momentum of the pulse is apparent in the momentum density (arrows).

large ka, the pulses have so much internal structure that they become difficult to portray). Even with m and ka fixed, the densities still depend on x, y, z, and t (or ρ, ϕ, z and t). We shall just show just two snapshots, each at $kct = 0$ and 2, of the energy and momentum densities in the zx plane (figure 1), in the zy plane (figure 2) and in the xy plane (figure 3). A three-dimensional picture of the energy density isosurface $e = \frac{1}{2}e_{\max}$ is shown in figure 4.

In figures 1(a), 2(a), 3(a) and 4, the snapshot is at time zero, when the pulse is mostly tightly localized in its focal region, centred on the origin. In figures 1(b), 2(b) and 3(b), the snapshot is taken at $kct = 2$, when the pulse has moved out of the focal region, predominantly

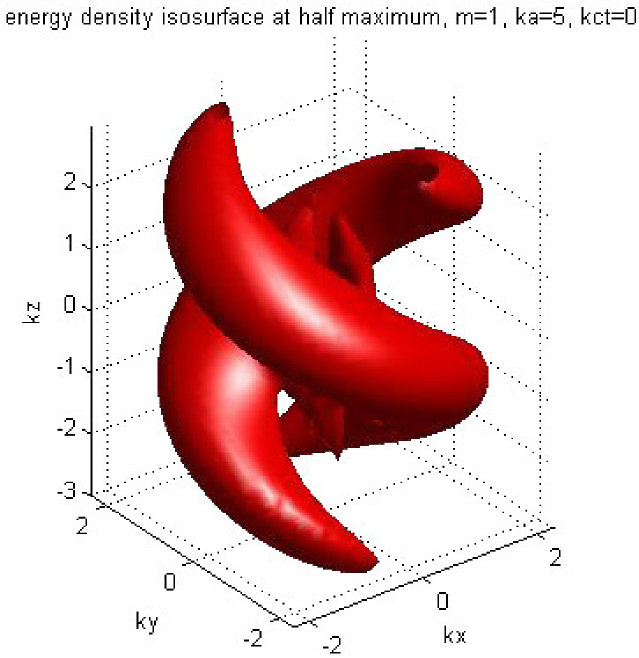

Figure 4. A three-dimensional view of the second-order energy density of the pulse, seen at time zero. The surfaces shown give the location of the half-maximum value of the energy density.

in the direction of positive z, spreading as it goes. Figures 1 and 2 show the pulse side-on to its propagation direction, while figure 3 shows the transverse motion.

The helical nature of the pulse is apparent in figure 3(b), but is most clear in the energy isosurface of figure 4. A similar helical structure can be found in electromagnetic pulses [21].

6. Summary

The total angular momentum $\mathbf{J}$ of an acoustic pulse is independent of time when viscosity and scattering are neglected. The component of $\mathbf{J}$ along the net direction of propagation (i.e. parallel to the total momentum $\mathbf{P}$ of the pulse) is independent of choice of origin. When $\mathbf{P}$ is along the z-axis, and ϕ is the azimuthal angle measuring rotation about the z-axis, the velocity potential $V(\mathbf{r}, t)$ must have ϕ-dependence for J_z to be non-zero. A set of solutions of the wave equation having azimuthal dependence is used to derive exact total energies, momenta, and angular momenta of a family of acoustic pulses. This family is characterized by a length a, a wavenumber k and a velocity potential magnitude V_0. In the limit of large ka, the energy, momentum and angular momentum may be interpreted in terms of the conventional phonon picture (energy $\hbar ck$, momentum $\hbar k$), augmented by the association of angular momentum $\hbar m$ with each phonon.

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