INSTITUTE OF PHYSICS PUBLISHING — JOURNAL OF OPTICS A: PURE AND APPLIED OPTICS

J. Opt. A: Pure Appl. Opt. **6** (2004) S128–S133 — PII: S1464-4258(04)66839-5

Angular momentum of electromagnetic pulses

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Received 31 July 2003, accepted for publication 29 October 2003
Published 24 February 2004
Online at stacks.iop.org/JOptA/6/S128 (DOI: 10.1088/1464-4258/6/3/021)

Abstract
Electromagnetic pulses are shown to have an intrinsic angular momentum along the direction of the net momentum of the pulse, unchanged by Lorentz boosts along this direction, and invariant to change of spatial origin. This intrinsic angular momentum is evaluated (together with the energy and momentum) for two types of pulses based on a localized oscillatory solution of the wave equation. The Lorentz boost required to bring each pulse to its zero-momentum frame is evaluated analytically.

Keywords: angular momentum, electromagnetic pulses, Lorentz transformations

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Plane wave pulses of the d'Alembert form $f(z-ct)$ predate electrodynamics, but it is only recently that solutions of the wave equation which are localized in three dimensions have been found and analysed [1–10]. Feng *et al* [5] evaluated the energy of an electromagnetic pulse based on the Ziolkowski solution of the wave equation [1, 2, 9], and the author evaluated the energy, momentum and angular momentum of several such pulses [11]. In each case, the energy U was greater than c times the momentum P_z of the pulse. This implies that a transformation to a zero-momentum Lorentz frame is possible. The author has recently shown that all pulse solutions of Maxwell's equations which are localized in space and time have energy greater than c times their momentum [12]. Thus all localized electromagnetic pulses have a zero-momentum frame; this fact is important for the existence of an intrinsic angular momentum of such pulses, as we show in the next section. Section 3 considers the energy, momentum and angular momentum of generic 'circularly polarized' ('CP') pulses; section 4 evaluates these for a new set of oscillatory localized wavefunctions. Section 5 then shows that another class of pulses has identically zero angular momentum.

2. Angular momentum of a pulse

The energy and momentum densities of an electromagnetic wave are given by [13]

$$u=\frac{1}{8\pi}(E^2+B^2),\qquad \boldsymbol{p}=\frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B}. \tag{1}$$

Consequently, the wave has an angular momentum density

$$\boldsymbol{j}(\boldsymbol{r},t)=\boldsymbol{r}\times\boldsymbol{p}(\boldsymbol{r},t)=\frac{1}{4\pi c}\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}). \tag{2}$$

In this paper we shall consider pulses localized in space and time, for which the total energy, momentum and angular momentum are all finite. These are given by

$$U=\int d^3r\,u(\boldsymbol{r},t),\qquad \boldsymbol{P}=\int d^3r\,\boldsymbol{p}(\boldsymbol{r},t),\qquad \boldsymbol{J}=\int d^3r\,\boldsymbol{j}(\boldsymbol{r},t). \tag{3}$$

As expected, the total energy and momentum of a pulse are independent of time [11]. The total angular momentum of a pulse is likewise a constant of the motion: from the Maxwell curl equations in vacuum, $\nabla\times\boldsymbol{E}+\partial_t\boldsymbol{B}=0$, $\nabla\times\boldsymbol{B}-\partial_t\boldsymbol{E}=0$ [∂_t stands for $\partial/\partial(ct)$] we have $\partial_t(\boldsymbol{E}\times\boldsymbol{B})=(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}-\frac{1}{2}\nabla(E^2+B^2)$, on using $\boldsymbol{E}\times(\nabla\times\boldsymbol{E})=\frac{1}{2}\nabla E^2-(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}$. Thus the electric contribution to $\partial_t J_z$ is given by

$$4\pi c\partial_t J_z^E=\int d^3r\,\{x[(E_x\partial_x+E_z\partial_z)E_y-\tfrac{1}{2}\partial_y(E_x^2+E_z^2)] - y[(E_y\partial_y+E_z\partial_z)E_x-\tfrac{1}{2}\partial_x(E_y^2+E_z^2)]\}. \tag{4}$$

The terms like $x\partial_y(E_x^2+E_z^2)$ integrate to zero, since the fields tend to zero at infinity. In the remaining terms we have, for example,

$$\int \mathrm{d}^3 r\, x(E_x\partial_x + E_z\partial_z)E_y$$
$$= -\int \mathrm{d}^3 r\, [xE_y(\partial_x E_x + \partial_z E_z) + E_x E_y]$$
$$= \int \mathrm{d}^3 r\, [xE_y(\partial_y E_y) - E_x E_y] = -\int \mathrm{d}^3 r\, E_x E_y \qquad (5)$$

on integrating by parts, using $\nabla \cdot \boldsymbol{E} = 0$, and integrating by parts again. The other non-zero part in (4) gives the negative of (5), and of course the same cancellation occurs in the magnetic parts of $\partial_t J_z$, and in the other two components of $\partial_t \boldsymbol{J}$. We can thus evaluate the angular momentum of the pulse at any finite time, as we can the energy U and momentum $\boldsymbol{P}$.

The energy and momentum values are independent of the choice of origin of the spatial coordinates, but the angular momentum is not: when $\boldsymbol{r} \to \boldsymbol{r} - \boldsymbol{a}$, $\boldsymbol{J} \to \boldsymbol{J} - \boldsymbol{a} \times \boldsymbol{P}$. Textbooks make statements like [14, p 569] 'the photon has vanishing mass and cannot be brought to rest in any Lorentz frame of reference', and so it would seem that we cannot associate an intrinsic angular momentum with an electromagnetic pulse. But it has been shown in [11] that pulses exist which have a zero-momentum frame (not a *rest* frame), and more recently that all localized electromagnetic pulses have energy greater than c times momentum, and thus that a Lorentz transformation to a zero-momentum frame is always possible [12]. Hence we can find the intrinsic angular momentum of a localized electromagnetic pulse by evaluating $\boldsymbol{J}$ in its zero-momentum frame. Furthermore, the component of $\boldsymbol{J}$ along its momentum vector in the laboratory frame (J_z, say) remains unchanged as we Lorentz-boost along the z-direction, to the zero-momentum frame or to any other inertial frame. This is because the four-tensor of angular momentum $J_{ij} = X_i P_j - X_j P_i$ (X_i and P_i represent components of the space–time and momentum–energy four-vectors) has the same structure as the electromagnetic field four-tensor composed of $\boldsymbol{E}$ and $\boldsymbol{B}$, with J_z corresponding to B_z (see for example [15, sections 2–6]). Since B_z does not change in a Lorentz boost along z, J_z will not either. We can thus evaluate the component of $\boldsymbol{J}$ along the direction of $\boldsymbol{P}$ in any Lorentz frame, and regard this unique value as the intrinsic angular momentum of the pulse. For the pulses we shall consider in this paper the other two components of angular momentum (J_x and J_y) will be identically zero, in any frame.

Most readers will be familiar with plane-wave pulses, based on solutions of the wave equation of the form $f(z - ct)$. Maxwell's equations may be satisfied by these transversely unbounded pulses, with $\boldsymbol{E}$ and $\boldsymbol{B}$ purely transverse. But the angular momentum density along z is given by

$$4\pi c j_z = [\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})]_z = E_z(xB_x + yB_y) - B_z(xE_x + yE_y) \qquad (6)$$

and hence if the propagation is along z and the fields are purely transverse, j_z will be zero everywhere. Here, in contrast, we deal with pulses bounded in all three dimensions. Such pulses are not purely transverse, and have non-zero j_z. They *can* have zero J_z, as we shall see.

3. 'Circularly polarized' pulses

A coherent monochromatic beam has a well-defined polarization at each point in space, and one can define a scalar function $\Lambda(\boldsymbol{r}) = |\boldsymbol{E}^2(\boldsymbol{r})|/|\boldsymbol{E}(\boldsymbol{r})|^2$ which gives the degree of linear polarization of the electric field [16]: $\Lambda \to 1$ for perfect linear polarization, and $\Lambda \to 0$ for perfect circular polarization. (Here $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_\mathrm{r}(\boldsymbol{r}) + \mathrm{i}\boldsymbol{E}_\mathrm{i}(\boldsymbol{r})$ is the complex field, and the time-dependent field is $\mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t}\} = \boldsymbol{E}_\mathrm{r}\cos\omega t + \boldsymbol{E}_\mathrm{i}\sin\omega t$.) As discussed in [16], $\Lambda(\boldsymbol{r})$ is related the local values of the Stokes parameters $s0$ and $s3$ [17]. For pulses we do not have polarization in the same sense: the electric and magnetic field vectors do not periodically orbit elliptical paths at a given point in space, as they do in a monochromatic beam. Strictly, we should (and shall) talk of the angular momentum of a pulse along the direction of its total momentum, which as we have seen is a well-defined invariant quantity. Nevertheless, beam polarization has proved a useful analogy in the construction of pulses with non-zero angular momentum. In [11] we considered a pulse with vector potential and fields given by

$$\boldsymbol{A} = \nabla \times [\mathrm{i}\psi, \psi, 0], \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} + \mathrm{i}\partial_t \boldsymbol{A},$$
$$\boldsymbol{E} = -\partial_t \boldsymbol{A} + \mathrm{i}\nabla \times \boldsymbol{A} \qquad (7)$$

where ψ is any solution of the wave equation $\nabla^2\psi = \partial_t^2\psi$. The particular solution of the wave equation used in [11], given explicitly in (16), is characterized by positive lengths a, b and magnitude ψ_0. It led to the energy, momentum and angular momentum values

$$U = \frac{\pi}{8}\frac{a+3b}{a^2}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-3b}{a^2}\psi_0^2,$$
$$cJ_z = \frac{\pi}{4}\frac{b}{a}\psi_0^2. \qquad (8)$$

Although U and P_z change in a Lorentz boost along z (for example the transformed momentum can be made zero by the boost with $\beta = cP_z/U = (a-3b)/(a+3b)$), the value of J_z is unchanged, as we have shown. In this section we consider some general properties of pulses constructed according to (7).

First we note that, for all pulses and beams for which the complex fields satisfy $\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{i}\boldsymbol{B}(\boldsymbol{r}, t)$, the energy and momentum densities, calculated for either the real or imaginary parts of the complex fields $\boldsymbol{E}$ and $\boldsymbol{B}$, are given by

$$u = \frac{1}{8\pi}|\boldsymbol{E}|^2 = \frac{1}{8\pi}|\boldsymbol{B}|^2, \qquad c\boldsymbol{p} = \frac{\mathrm{i}}{8\pi}\boldsymbol{E}\times\boldsymbol{E}^* = \frac{\mathrm{i}}{8\pi}\boldsymbol{B}\times\boldsymbol{B}^*. \qquad (9)$$

Specializing to the 'CP' pulse specified by (7), the magnetic field is

$$\boldsymbol{B} = [(\partial_x - \mathrm{i}\partial_y)\partial_y - \mathrm{i}(\partial_z + \partial_t)\partial_z, -(\partial_x - \mathrm{i}\partial_y)\partial_x$$
$$- (\partial_z + \partial_t)\partial_z, \mathrm{i}(\partial_x - \mathrm{i}\partial_y)(\partial_z + \partial_t)]\psi. \qquad (10)$$

When $\psi(\boldsymbol{r}, t)$ is independent of the azimuthal angle ϕ we use the cylindrical polar coordinates $\rho = (x^2 + y^2)^{\frac{1}{2}}$ and z; with $\psi = \psi(\rho, z, t)$ the energy density is given by

$$8\pi u = 2|\partial_z(\partial_z + \partial_t)\psi|^2 + |\partial_\rho^2\psi|^2 + \rho^{-2}|\partial_\rho\psi|^2$$
$$+ |\partial_\rho(\partial_z + \partial_t)\psi|^2 + 2\,\mathrm{Re}\{(\partial_\rho^2 + \rho^{-1}\partial_\rho)\psi \cdot (\partial_z + \partial_t)\partial_z\psi^*\}. \qquad (11)$$

There are three components of the momentum density, of which we find convenient to express p_x and p_y in terms of the radial and azimuthal momentum densities p_ρ and p_ϕ:

$$p_x = p_\rho\cos\phi - p_\phi\sin\phi, \qquad p_y = p_\rho\sin\phi + p_\phi\cos\phi. \qquad (12)$$

This is because (as expected) j_z is determined purely by the azimuthal momentum density,

$$j_z = xp_y - yp_x = \rho p_\phi \tag{13}$$

and also because p_ρ and p_ϕ are independent of ϕ when ψ is. For the class of pulses specified in (7) we have

$$\begin{aligned}
-4\pi c p_z &= |\partial_z(\partial_z + \partial_t)\psi|^2 \\
&\quad + \mathrm{Re}\{(\partial_\rho^2 + \rho^{-1}\partial_\rho)\psi \cdot \partial_z(\partial_z + \partial_t)\psi^* + \rho^{-1}\partial_\rho\psi \cdot \partial_\rho^2\psi^*\} \\
-4\pi c p_\rho &= \mathrm{Re}\{[\partial_\rho^2 + \partial_z(\partial_z + \partial_t)]\psi \cdot \partial_\rho(\partial_z + \partial_t)\psi^*\} \\
-4\pi c p_\phi &= \mathrm{Im}\{[\rho^{-1}\partial_\rho + \partial_z(\partial_z + \partial_t)]\psi \cdot \partial_\rho(\partial_z + \partial_t)\psi^*\}.
\end{aligned} \tag{14}$$

Note that ψ has to be complex for non-zero p_ϕ, and so for non-zero J_z. When ψ is independent of the azimuthal angle, all of u, p_z, p_ρ and p_ϕ are also. Hence the transverse components of the angular momentum density,

$$\begin{aligned}
j_x &= yp_z - zp_y = \rho p_z \sin\phi - z[p_\rho \sin\phi + p_\phi \cos\phi] \\
j_y &= zp_x - xp_z = z[p_\rho \cos\phi - p_\phi \sin\phi] - \rho p_z \cos\phi
\end{aligned} \tag{15}$$

are linear in $\sin\phi$ and $\cos\phi$ and integrate to zero over the azimuthal angle. J_z is thus the only non-zero component of angular momentum for this class of pulses.

The above results apply to 'CP' pulses constructed from all possible localized solutions of the wave equation [1–10]. The energy, momentum and angular momentum values quoted in (8) correspond to the Ziolkowski wavefunction [1]

$$\psi(\boldsymbol{r}, t) = \frac{ab}{\rho^2 + [a - \mathrm{i}(z + ct)][b + \mathrm{i}(z - ct)]}\psi_0. \tag{16}$$

This has no oscillations. In the next section we investigate 'CP' pulses based on an oscillatory solution which reduces to (16) when the wavenumber k goes to zero.

4. 'CP' pulse based on an oscillatory wavefunction

The Ziolkowski wavefunction (16) can be obtained from his more general solution

$$\psi(\boldsymbol{r}, t) = \int_0^\infty \mathrm{d}k\, F(k) \frac{\exp\{\mathrm{i}k(z + ct) - k\rho^2/[b + \mathrm{i}(z - ct)]\}}{b + \mathrm{i}(z - ct)} \tag{17}$$

by setting $F(k)$ proportional to e^{-ka}. For oscillatory solutions which (over a finite region) approximate the plane wave $\exp \mathrm{i}k(z - ct)$, we time-reverse (17), and write it in the form

$$\psi(\boldsymbol{r}, t) = [b + \mathrm{i}(z + ct)]^{-1} \int_0^\infty \mathrm{d}k\, F(k)\mathrm{e}^{-ks}, \qquad s = \frac{\rho^2}{b + \mathrm{i}(z + ct)} - \mathrm{i}(z - ct). \tag{18}$$

The integral in (18) is some function $f(s)$ for each chosen $F(k)$. We see that

$$\psi(\boldsymbol{r}, t) = \frac{f(s)}{b + \mathrm{i}(z + ct)} \tag{19}$$

is therefore a solution of the wave equation, for any f. This result was first obtained by Hillion [3], by another route.

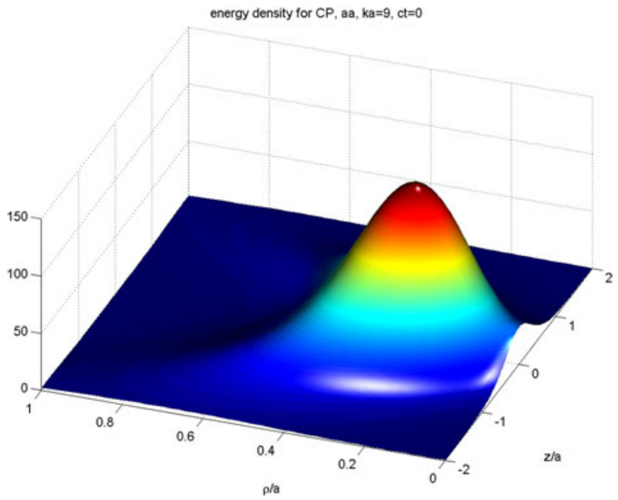

Figure 1. Energy density plot for the 'CP' pulse of section 3, evaluated for the wavefunction (22) with $ka = 9$. The figure is drawn for $t = 0$, at which time the pulse is concentrated near the origin in its focal region. Note that the transverse scale ρ/a is stretched in comparison to the longitudinal scale z/a.

We now consider the special case $f(s) = \mathrm{e}^{-ks}/(s + a)$:

$$\begin{aligned}
\psi(\boldsymbol{r}, t) &= \frac{ab\mathrm{e}^{-ks}}{(s + a)[b + \mathrm{i}(z + ct)]}\psi_0 \\
&= \frac{ab\mathrm{e}^{-ks}}{\rho^2 + [a - \mathrm{i}(z - ct)][b + \mathrm{i}(z + ct)]}\psi_0
\end{aligned} \tag{20}$$

(a and b are positive lengths, and k is a positive wavenumber). When $k \to 0$ this reduces to the time-reversed form of (16). When a and b are large compared to $|z \pm ct|$, and also b is large compared to $k\rho^2$, the wave function tends to $\mathrm{e}^{\mathrm{i}k(z-ct)}$. The absolute square of ψ is

$$|\psi|^2 = \frac{a^2b^2 \exp\{-2kb\rho^2/[b^2 + (z + ct)^2]\}}{[\rho^2 + ab + z^2 - c^2t^2]^2 + [a(z + ct) - b(z - ct)]^2}\psi_0^2. \tag{21}$$

This suggests that pulses constructed from ψ will have forward and backward propagating parts. At given time t, the spatially smaller part will be at $z = -ct$, since the exponent then becomes $-2k\rho^2/b$. At $z = ct$ the exponent is smaller in magnitude, namely $-2kb\rho^2/[b^2 + 4c^2t^2]$. The propagation of electromagnetic pulses based on (20) is overwhelmingly in the forward direction when ka and kb are large.

From now on we specialize to the case $a = b$, since we are then able to evaluate the energy, momentum and angular momentum integrals analytically. We thus work with

$$\psi(\boldsymbol{r}, t) = \frac{a^2\mathrm{e}^{-ks}}{\rho^2 + [a - \mathrm{i}(z - ct)][a + \mathrm{i}(z + ct)]}\psi_0, \qquad s = \frac{\rho^2}{a + \mathrm{i}(z + ct)} - \mathrm{i}(z - ct). \tag{22}$$

This wavefunction is characterized by an amplitude ψ_0 and by a single dimensionless parameter, ka. The energy density for the 'CP' pulse based on (22) is shown in figures 1 and 2 for $ct = 0$ and $2a$; the angular momentum density at $ct = 2a$ is shown in figure 3. Since the energy, momentum and angular momentum of a pulse are all independent of time, we can evaluate the integrals at $t = 0$. The evaluation of the integrals (3) proceeds

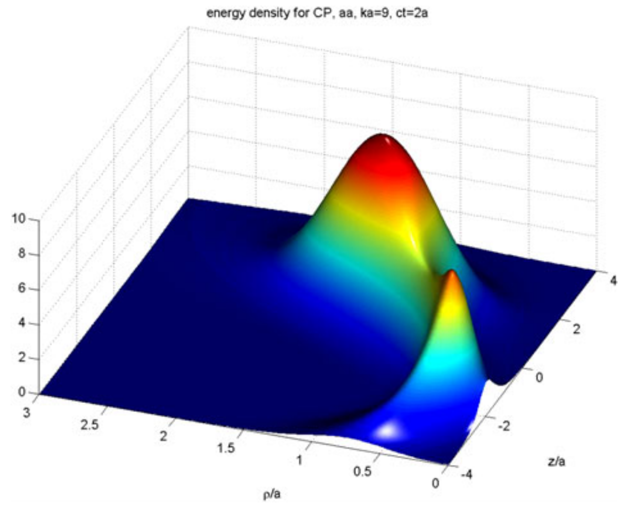

Figure 2. As for figure 1, but at time $t = 2a/c$. The pulse has now spread from the focal region in both the forward and backward directions, with both energy and momentum being concentrated in an annulus moving in the positive z direction. Note the vertical scale: the energy density has decreased more than tenfold from the $t = 0$ value in figure 1, and has spread in space. (The total energy remains the same, of course.) Note also the different horizontal scales.

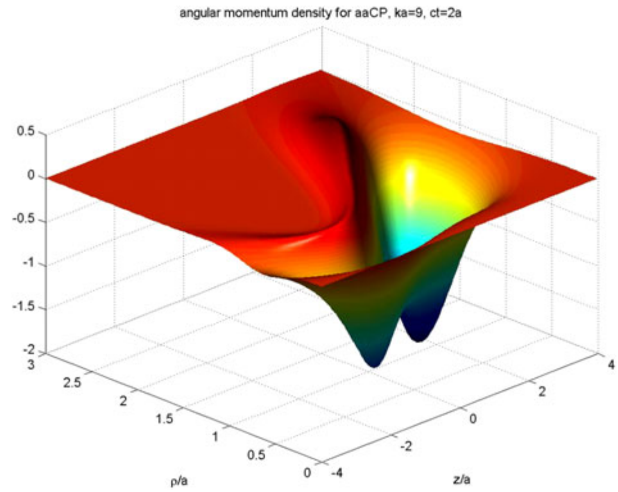

Figure 3. The angular momentum density $j_z = \rho p_\phi$ of the 'CP' pulse shown in figure 2 at $ct = 2a$. The angular momentum density is predominantly negative at all times, changing form as the pulse propagates, but always giving $J_z = \int d^3r\, j_z = \frac{-\pi}{4c}\psi_0^2$ (equation (23)).

as in [11] on substitution of (22) into (11) and (14). Exponential integrals Ei appear at the intermediate stage, but when $a = b$ the final results are simple:

$$U = \frac{\pi}{4a}(ka+2)\psi_0^2, \qquad cP_z = \frac{\pi}{4a}(ka-1)\psi_0^2, \qquad cJ_z = -\frac{\pi}{4}\psi_0^2. \tag{23}$$

When $ka \to 0$ these results are in accord with those obtained by setting $b = a$ in (8), namely $U = \frac{\pi}{2a}\psi_0^2$, $cP_z = -\frac{\pi}{4a}\psi_0^2$, $cJ_z = \frac{\pi}{4}\psi_0^2$ (time-reversal of (16) when $a = b$ is equivalent to complex conjugation of ψ, which changes the sign of p_ϕ and hence of the angular momentum).

We see from (23) that a Lorentz boost at speed βc, with

$$\beta = \frac{cP_z}{U} = \frac{ka-1}{ka+2} \tag{24}$$

will put us into the zero-momentum frame of the pulse. In this frame the energy is

$$U_0 = \gamma(U - \beta cP_z) = U/\gamma, \qquad \gamma = (1-\beta^2)^{-\frac{1}{2}}. \tag{25}$$

We can associate an angular frequency ω with the pulse as we did in [11] by setting $U = \hbar\omega$ and $|J_z| = \hbar$, so

$$\omega = U/|J_z| = c(k + 2/a). \tag{26}$$

As we expect, $\omega \to ck$ for $ka \gg 1$. The frequency in the zero-momentum frame is $\omega_0 = \omega/\gamma$, whereas we might have expected the Doppler result

$$\frac{\omega}{\omega_0} = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \text{(Doppler shift for monochromatic plane wave)}, \tag{27}$$

at least in the limit $ka \gg 1$. The reason for the $\omega = \gamma\omega_0$ result as opposed to (27) appears to be that in the zero-momentum frame there is equal amount of propagation in the $+z$ and $-z$ directions. On boosting to the laboratory frame, there is both a blue shift and a red shift, with the average shift being

$$\frac{1}{2}\left\{\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}}\right\} = \frac{1}{\sqrt{1-\beta^2}} = \gamma. \tag{28}$$

(The author is grateful to Damien Martin and Matt Visser for discussions clarifying this point.)

The expression for the azimuthal momentum density in (14) is of the form

$$p_\phi = \mathrm{Im}\{(C\psi)D\psi^*\} \tag{29}$$

where C and D are real differential operators. As we noted below (14), ψ has to be complex for this to be non-zero. We can change the sign of p_ϕ (and thus of J_z) by interchanging ψ and ψ^*. We can think of the respective pulses as having opposite circular polarization, loosely speaking. If we combine these in equal amounts, i.e. take the wavefunction to be $\psi \pm \psi^*$, the resulting p_ϕ and angular momentum become zero. This corresponds to the combination of left and right circular polarizations to obtain linearly polarized light, which of course has no angular momentum.

5. The TE + iTM pulse

The TE + iTM pulse [11] is constructed by coherent superposition of transverse magnetic (TM) and transverse electric (TE) pulses; it has the fields

$$\boldsymbol{A} = \nabla \times [0, 0, \psi], \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} + \mathrm{i}\partial_t \boldsymbol{A}, \qquad \boldsymbol{E} = \mathrm{i}\boldsymbol{B} \tag{30}$$

(ψ is again any pulse solution of the wave equation). The energy and momentum densities are given by (9), as they are for all $\boldsymbol{E} = \pm\mathrm{i}\boldsymbol{B}$ pulses or beams. For ψ independent of the

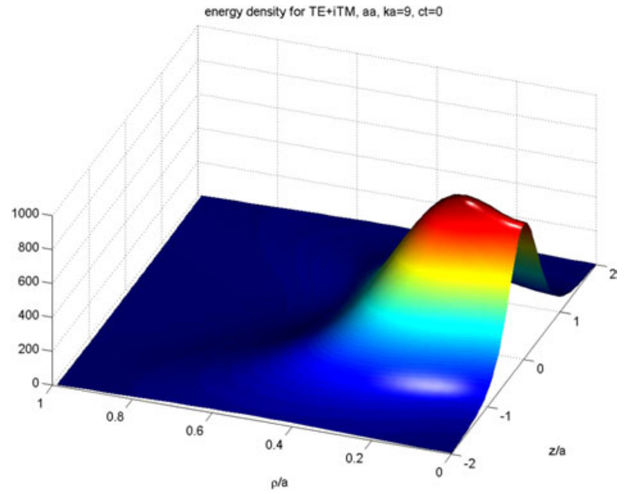

Figure 4. Energy density plot for the TE + iTM pulse of section 5, evaluated for the wavefunction (22) with $ka = 9$. The figure is drawn for $t = 0$, at which time the pulse is concentrated near the origin in its focal region. Note the transverse scale ρ/a is stretched relative to the longitudinal scale z/a.

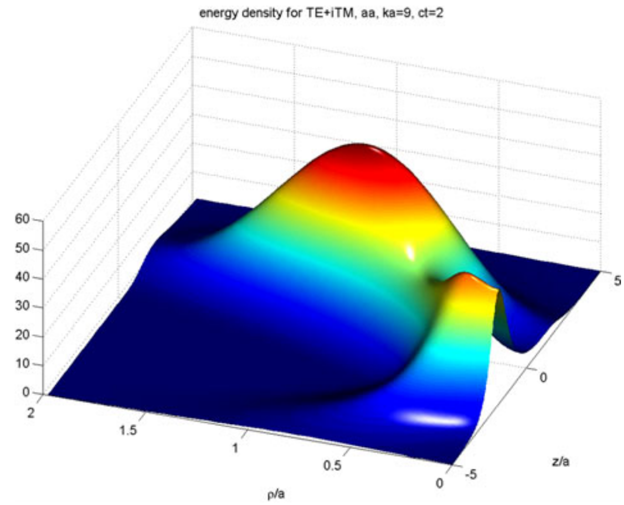

Figure 5. As for figure 4, now at time $t = 2a/c$. Note both forward and backward progression, but with the motion being predominantly forward. As for the 'CP' pulse, the energy and momentum are concentrated in a forward-moving annulus. Note the vertical (energy density) scale in relation to that of figure 4: the energy density has decreased in magnitude, and has spread.

azimuthal angle we find [11]

$$\begin{aligned}
8\pi u &= |\partial_\rho\partial_z\psi|^2 + |\partial_\rho\partial_t\psi|^2 + |(\partial_z^2 - \partial_t^2)\psi|^2 \\
-4\pi c p_z &= \mathrm{Re}\{(\partial_\rho\partial_t\psi^*)(\partial_\rho\partial_z\psi)\} \\
4\pi c p_\rho &= \mathrm{Re}\{(\partial_\rho\partial_t\psi^*)(\partial_z^2 - \partial_t^2)\psi\} \\
4\pi c p_\phi &= \mathrm{Im}\{(\partial_\rho\partial_z\psi^*)(\partial_z^2 - \partial_t^2)\psi\}.
\end{aligned} \tag{31}$$

As for the 'CP' pulse the azimuthal momentum density p_ϕ and consequently the angular momentum density $j_z = \rho p_\phi$ are zero unless ψ is complex. The energy density for the TE + iTM pulse based on the wavefunction (22) is shown in figures 4 and 5 for $ct = 0$ and $2a$; the angular momentum density at $ct = 2a$ is shown in figure 6. In [11] the total energy, momentum and angular momentum were evaluated for the TE + iTM electromagnetic pulse corresponding to the wavefunction (16). We found

$$U = \frac{\pi}{8}\frac{a+b}{ab}\psi_0^2, \qquad cP_z = \frac{\pi}{8}\frac{a-b}{ab}\psi_0^2, \qquad J_z = 0. \tag{32}$$

Here we shall just give the results corresponding to the oscillatory wavefunction (22):

$$U = \frac{\pi}{4a}(ka+1)^2\psi_0^2, \qquad cP_z = \frac{\pi}{4a}(ka)^2\psi_0^2, \qquad J_z = 0. \tag{33}$$

The $k \to 0$ limits of (33) agree with the $a = b$ values of (32), as they must. The Lorentz boost speed βc along the z-axis required to bring the pulse to its zero-momentum frame is given by

$$\beta = \frac{cP_z}{U} = \left(\frac{ka}{ka+1}\right)^2. \tag{34}$$

The fact that $J_z = 0$ for TE + iTM pulses follows from ψ being independent of the azimuthal angle, and satisfying the wave equation: we have $(\partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2 - \partial_t^2)\psi = 0$, so from (13) and (31) J_z is given by

$$4\pi c J_z = -\int d^3r\ \mathrm{Im}\{(\partial_\rho\partial_z\psi^*)\rho^{-1}\partial_\rho(\rho\partial_\rho\psi)\}. \tag{35}$$

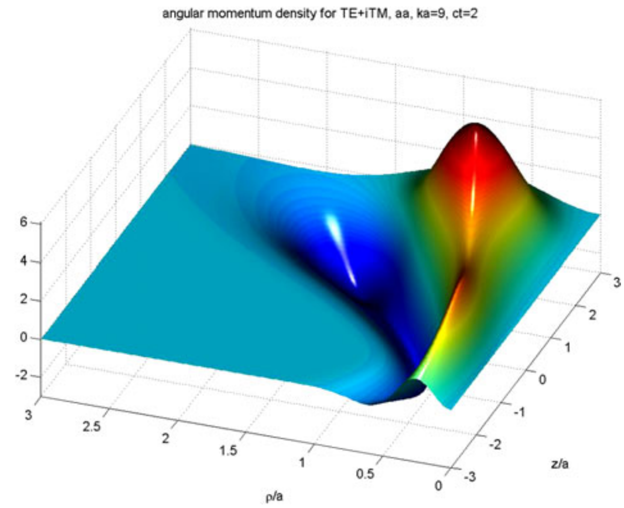

Figure 6. The angular momentum density $j_z = \rho p_\phi$ of the TE + iTM pulse shown in figure 5 at $ct = 2a$. The angular momentum density has positive and negative regions. These cancel at all times, making $J_z = \int d^3r\ j_z$ zero.

Time-reversal changes the sign of J_z. But (35) contains no time derivatives, and can be evaluated at $t = 0$, so time-reversal does not change its value. Hence J_z must be zero.

6. Discussion

We have shown that the angular momentum of an electromagnetic pulse is a constant of the motion (as expected). For a pulse with net momentum in the z-direction, J_z is the intrinsic angular momentum of the pulse, unchanged by Lorentz boosts along z, and invariant to change of spatial origin. For the pulses considered here, the other two components are zero, in any frame. We have evaluated the energy, momentum and angular momentum for two types of pulses using a new exact oscillatory solution of the wave equation. For the 'CP' pulse we regain the plane wave relation $\omega = U/|J_z|$ in the $ka \gg 1$ limit.

The pulse types considered here are based on wavefunctions which are independent of the azimuthal angle. The angular momentum properties come from the twist in the fields, not in the wavefunction ψ. When ψ has ϕ-dependence, for example of the form $e^{\pm im\phi}$ with m an integer, one may expect new effects to appear, leading to what is known as 'orbital' angular momentum. Allen *et al* [18] have reviewed such phenomena for light beams, and we intend to explore them for light pulses in another publication. However, the fact that J_z is invariant to Lorentz boosts along the momentum direction of the pulse and is independent of the choice of origin is a general result, valid for any pulse solution of Maxwell's equations, and is thus not altered by ϕ-dependence of the wavefunction.

Acknowledgments

The author is grateful to Damien Martin, Matt Visser and the anonymous referees for constructive comments.

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