IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **30** (2009) L43–L46 doi:10.1088/0143-0807/30/3/L04

LETTERS AND COMMENTS

Airy wavepacket solutions of the Schrödinger equation

John Lekner

The MacDiarmid Institute for Advanced Materials and Nanotechnology, and School of Chemical and Physical Sciences, Victoria University of Wellington, P O Box 600, Wellington, New Zealand

Received 27 January 2009, in final form 10 March 2009
Published 7 April 2009
Online at stacks.iop.org/EJP/30/L43

Abstract
A Galilean boost to a frame moving with a complex velocity transforms the Berry–Balazs Airy function solution of the time-dependent Schrödinger equation into a square-integrable wavepacket. This packet has fixed momentum and energy; it contracts as time increases to zero, and then spreads again. The Berry–Balazs solution is revealed as a limiting form of a well-behaved free particle wavepacket. The topic is suitable for graduate or advanced undergraduate quantum mechanics courses.

Berry and Balazs [1] considered a solution of the Schrödinger time-dependent equation for one-dimensional free motion of a free particle of mass m,

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi, \tag{1}$$

namely

$$\psi_{BB}(x,t) = Ai\left[q\left(x - \tfrac{1}{2}at^2\right)\right] e^{i\frac{mat}{\hbar}[x-\frac{1}{3}at^2]}. \tag{2}$$

Here q is a real wavenumber, defined in terms of the 'acceleration' a by

$$q^3 = \frac{2m^2a}{\hbar^2} \qquad (a > 0). \tag{3}$$

Their assertion that 'the probability density $|\psi|^2$ propagates in free space without distortion and with constant acceleration' produced some interesting interpretations and generalizations [2–6].

However, this letter shows that the Berry–Balazs solution is a limiting form of a free particle wavepacket which (as is normal) converges to a focal region and then spreads out again, and which does not accelerate. Thus the paradoxical 'acceleration without external force' aspect of the Berry–Balazs wavepacket is removed. Likewise, the 'propagation without distortion', which is impossible for normalizable free-space wavepackets (see the discussion of expectation values below, and particularly equation (18)), is seen to hold only in the non-square-integrable case. Put simply: it makes no physical sense to speak of position, or velocity, or acceleration, when their expectation values do not exist. Square integrability is essential in the quantum mechanics of wavepackets. Indeed, Berry and Balazs say as much in their paper,

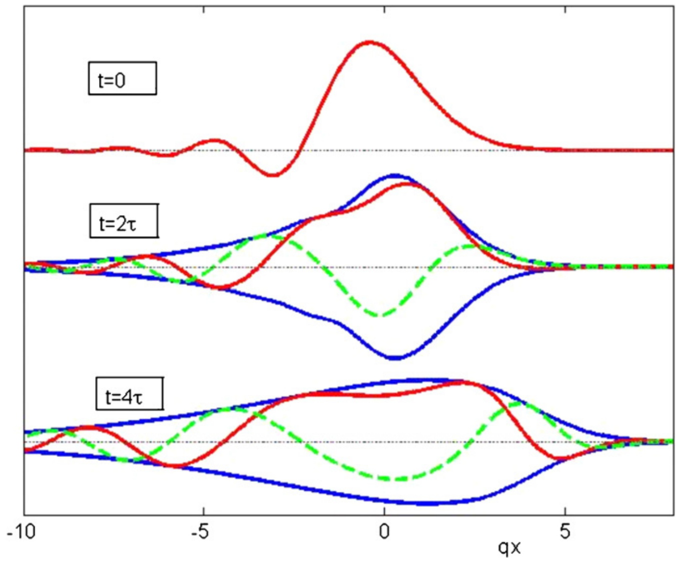

Figure 1. Spreading of the Airy packet (5) in its zero-momentum ($u = 0$) frame. The modulus $|\psi|$ and the real and imaginary parts of ψ are shown at three successive times: $t = 0, 2\tau$ and 4τ where $\tau = (\hbar/4ma^2)^{1/3}$. The top frame shows just the real part: the imaginary part is zero at $t = 0$. The unit of length is $q^{-1} = (\hbar^2/2m^2a)^{1/3}$, and v is set equal to $(\hbar a/4m)^{1/3}$. The imaginary part is shown dashed. The centre of probability remains fixed at $q\langle x\rangle = -\frac{1}{4}$. Animations of this packet can be viewed at http://www.victoria.ac.nz/scps/staff/johnlekner/animations.aspx.

(This figure is in colour only in the electronic version)

and stress that what accelerates in (2) is the position where the argument of the Airy function is zero, not an expectation value of position.

The function ψ_{BB} is not square integrable, since the Airy function $Ai(z)$, which is the regular solution of $\mathrm{d}^2\psi/\mathrm{d}z^2 = z\psi$, has the asymptotic forms [7, 8] (with $\zeta = \frac{2}{3}z^{3/2}$)

$$\begin{aligned} Ai(z) &\to \frac{1}{2\sqrt{\pi}}z^{-1/4}\exp(-\zeta) \qquad |\arg(z)| < \pi \\ Ai(-z) &\to \frac{1}{\sqrt{\pi}}z^{-1/4}\sin(\zeta+\pi/4) \qquad |\arg(z)| < 2\pi/3. \end{aligned} \tag{4}$$

In this letter we shall discuss the properties of a square-integrable packet which is a generalization of (2), namely

$$\psi(x,t) = Ai\left[q\left(x - ut + \mathrm{i}vt - \tfrac{1}{2}at^2\right)\right] \mathrm{e}^{\mathrm{i}\frac{mat}{\hbar}[x-ut-\frac{1}{3}at^2]}\,\mathrm{e}^{\frac{mv}{\hbar}[x-ut+\frac{\mathrm{i}}{2}vt-at^2]}\,\mathrm{e}^{\mathrm{i}\frac{mu}{\hbar}[x-\frac{1}{2}ut]}. \tag{5}$$

As the time t increases from negative values the packet described by (5) contracts, reaching its most compact form at $t = 0$, and then spreads as time increases. Despite the appearance of (5), we shall show that there is no acceleration, for any u and for any $v > 0$. The time development of the wavepacket (5) is illustrated in figure 1.

The wavepacket (5) can be obtained from the Berry–Belazs wavefunction (2) by a complex-velocity Galilean transformation. In the real Galilean boost [9]

$$x \to x - ut, \qquad t \to t \tag{6}$$

the transformed wavefunction is augmented by the phase factor

$$\exp\left[\frac{\mathrm{i}mu}{\hbar}\left(x - \frac{1}{2}ut\right)\right]. \tag{7}$$

When ψ_{BB} is boosted to the complex speed $u - \mathrm{i}v$, the wavepacket (5) results. In the special case $u = 0$ we regain a solution equivalent to a recent paraxial approximation for planar optical beams [10].

We now consider the properties of the wavepacket (5). Direct differentiation shows that it is a solution of the free-particle Schrödinger equation (1). It is square integrable for $v > 0$, because of the $\mathrm{e}^{\frac{mv}{\hbar}x}$ factor. (The asymptotic forms given in (4) are applicable because $\arg\left[q\left(x - ut + \mathrm{i}vt - \frac{1}{2}at^2\right)\right] = \arctan[vt/(x - ut - \frac{1}{2}at^2)]$ lies between $-\pi/2$ and $\pi/2$).

That the normalization

$$N = \int_{-\infty}^{\infty} \mathrm{d}x\, |\psi(x,t)|^2 \tag{8}$$

is independent of time follows from the Schrödinger equation (as is shown in standard texts on quantum mechanics): on replacing time derivatives by space derivatives according to (1) we find

$$\partial_t N = \frac{\mathrm{i}\hbar}{2m} \int_{-\infty}^{\infty} \mathrm{d}x \partial_x (\psi^* \partial_x \psi - \psi \partial_x \psi^*) = 0. \tag{9}$$

(This is the global (integrated) form of the equation of continuity or conservation of probability.) At $t = 0$ we have $N = \int_{-\infty}^{\infty} \mathrm{d}x [Ai(qx)]^2\, \mathrm{e}^{2mvx/\hbar}$, and the integral representation [1, 7, 8]

$$Ai(qx) = \frac{1}{2\pi q} \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}(kx + k^3/3q^3)} \tag{10}$$

together with

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}(k - k' - \mathrm{i}Q)x} = 2\pi\delta(k - k' - \mathrm{i}Q) \tag{11}$$

gives us, on setting $Q = 2mv/\hbar$,

$$N = \frac{\exp\left(\frac{Q^3}{12q^3}\right)}{2\sqrt{\pi Qq}}. \tag{12}$$

The expectation values of $p = -\mathrm{i}\hbar\partial_x$, p^2, x and x^2 can all be found analytically. We also list the mean square deviations $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$ and $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$:

$$\langle p\rangle = mu, \qquad \frac{1}{2m}\langle p^2\rangle = \frac{\hbar a}{4v} + \frac{1}{2}mu^2, \qquad (\Delta p)^2 = \frac{\hbar m a}{2v} \tag{13}$$

$$\langle x\rangle = \frac{v^2}{2a} - \frac{\hbar}{4mv} + ut, \qquad (\Delta x)^2 = \frac{1}{8}\left(\frac{\hbar}{mv}\right)^2 + \frac{1}{2}\frac{\hbar v}{ma} + \frac{1}{2}\frac{\hbar a}{mv}t^2. \tag{14}$$

For comparison, we list the corresponding quantities for the free-space Gaussian packet [11–14]:

$$\Phi_0(x,t) = \frac{b}{\sqrt{b^2 + \frac{\mathrm{i}\hbar t}{m}}} \exp\left\{\mathrm{i}\frac{mu}{\hbar}\left(x - x_0 - \frac{1}{2}ut\right) - \frac{(x - x_0 - ut)^2}{2(b^2 + \mathrm{i}\hbar t/m)}\right\} \tag{15}$$

$$\langle p\rangle = mu, \qquad \frac{1}{2m}\langle p^2\rangle = \frac{\hbar^2}{4mb^2} + \frac{1}{2}mu^2, \qquad (\Delta p)^2 = \frac{\hbar^2}{2b^2} \tag{16}$$

$$\langle x\rangle = x_0 + ut, \qquad (\Delta x)^2 = \frac{1}{2}\left[b^2 + \left(\frac{\hbar t}{mb}\right)^2\right]. \tag{17}$$

We note that the positional mean square deviation is in each case in accord with the general result [15–17]

$$(\Delta x)^2 = (\Delta x)_0^2 + \left(\frac{\Delta p}{m}\right)^2 t^2. \qquad (18)$$

(We use the form valid when the spread is minimum at $t = 0$.)

It is also interesting to note the respective uncertainty products:

$$\text{Airy}: \quad (\Delta x \Delta p)^2 = \frac{\hbar^2}{4}\left[1 + \frac{1}{4}\frac{\hbar a}{m v^3} + \frac{a^2}{v^2}t^2\right] \qquad (19)$$

$$\text{Gaussian}: \quad (\Delta x \Delta p)^2 = \frac{\hbar^2}{4}\left[1 + \left(\frac{\hbar}{mb^2}\right)^2 t^2\right]. \qquad (20)$$

The Gaussian packet has the minimum value $\hbar/2$ of $\Delta x \Delta p$ at $t = 0$, whereas the Airy packet has $\Delta x \Delta p > \hbar/2$, since its square integrability depends on a positive value of v.

If we set $u = 0$ and let $v \to 0$ we regain the Berry–Balazs function (2). In this limit the norm diverges as $v^{-1/2}$, and the expectation values of energy and position also diverge. For any $v > 0$ the expectation value of the momentum is zero, and $\langle x \rangle$ is fixed at $v^2/2a - \hbar/4mv$ [see (14)]. The at^2 terms in (5) determine the shape and size of the packet, as well as its energy and momentum, and do not give an actual acceleration.

To sum up, particles in free space do not accelerate, and we have shown that the Berry–Balazs Airy function solution, which has at^2 terms in it that strongly suggest acceleration, is in fact a limiting case of a well-behaved, square-integrable family of wavepackets. The expectation values of position and momentum show no acceleration. All is normal except in the Berry–Balazs limit, where the solution is not square integrable.

Acknowledgment

The author is grateful to an anonymous reviewer for helpful comments.

References

[1] Berry M V and Balazs N L 1979 Nonspreading wave packets *Am. J. Phys.* **47** 264–7
[2] Greenberger D M 1980 Comment on 'Nonspreading wave packets' *Am. J. Phys.* **48** 256
[3] Holstein B R 1983 The extended Galilean transformation and the path integral *Am. J. Phys.* **51** 1015–6
[4] Nassar A B, Bassalo J M F and Alencar P S 1995 Dispersive Airy packets *Am. J. Phys.* **63** 849–52
[5] Unnikrishnan K and Rau A R P 1996 Uniqueness of the Airy packet in quantum mechanics *Am. J. Phys.* **64** 1034–5
[6] Woo C H 1997 Particles pushed by an accelerated mirror *Am. J. Phys.* **65** 446
[7] Antosiewicz H A 1964 Bessel functions of fractional order *Handbook of Mathematical Functions* (NBS Applied Math. Series No. 55) ed M Abramowitz and I A Stegun (Washington DC: US Government Printing Office) chapter 10
[8] Valleé O and Soares M 2004 *Airy Functions and Applications in Physics* (London: Imperial College Press)
[9] Ballentine L E 1998 *Quantum Mechanics: A Modern Development* (Singapore: World Scientific) section 4.3
[10] Siviloglou G A and Christodoulides D N 2007 Accelerating finite energy Airy beams *Opt. Lett.* **32** 979–81
[11] Kennard E H 1927 Zur quantenmechanik einfacher bewegungstypen *Z. Phys.* **44** 326–52
[12] Darwin C G 1928 Free motion in quantum mechanics *Proc. R. Soc.* A **117** 258–93
[13] Lekner J 2007 Reflectionless eigenstates of the sech^2 potential *Am. J. Phys.* **75** 1151–7
[14] Cox C and Lekner J 2008 Reflection and non-reflection of particle wavepackets *Eur. J. Phys.* **29** 671–9
[15] Nicola M 1972 Position uncertainty of a free particle: a matrix derivation *Am. J. Phys.* **40** 342
[16] Bradford H M 1976 Propagation and spreading of a pulse or wave packet *Am. J. Phys.* **44** 1058–63
[17] Klein J R 1980 Do free quantum-mechanical wavepackets always spread? *Am. J. Phys.* **48** 1035–7